SPHERICAL HARMONICS

AN ELEMENTARY TREATISE
ON
HARMONIC FUNCTIONS
WITH APPLICATIONS

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WITH TWENTY DIAGRAMS

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PREFACE

THE writing of this book was undertaken with the object of providing a text-book on the elements of the theory of the Spherical Harmonics, with applications to mathematical physics, so far as this could be done without employing the method of contour integration. Subsequently it was thought advantageous to include discussions on similar lines of Fourier Series and Bessel Functions, with corresponding applications.

The first chapter contains an elementary account of the theory of Fourier Series, while the second and third deal with the applications of Fourier Series to Conduction of Heat and Vibrations of Strings. The four following chapters form the central part of the book. In Chapter IV. the Spherical Harmonics are defined, and a summary is given of the elementary properties of the Hypergeometric Function. Chapters V., VI., and VII. are devoted respectively to the Legendre Coefficients, the Legendre Functions, and the Associated Legendre Functions.

In Chapters VIII., IX., and X. the Spherical Harmonics are employed to obtain expressions for the gravitational and electrostatic potentials of bodies bounded by circles, spheres, and spheroids; Chapters XI. and XII. include similar discussions for bodies bounded by ellipsoids of revolution and eccentric spheres. A short account of Clerk Maxwell's theory of the Spherical Harmonics will be found in Chapter XIII. The remaining three chapters deal with the Bessel Functions and their applications to Vibrations of Membranes and Conduction of Heat.

At all stages of the work, as in the course of many previous undertakings, I have been indebted to Professor G. A. Gibson, LL.D., for important criticisms and valuable suggestions. To him my warmest thanks are due. I have also to thank my colleague, Mr. William Arthur, M.A., for the great care with which he has read through all the proof sheets.

Among the books that proved useful to me, special mention should be made of the following: Wangerin's *Theorie des Potentials und der Kugelfunktionen*; Carslaw's *Conduction of Heat*; Schafheitlin's *Theorie der Besselschen Funktionen*; and Lamb's *Dynamical Theory of Sound*. I have also made use of lectures by Professor E. W. Hobson, F.R.S.

THOMAS M. MACROBERT

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CHAPTER I

FOURIER SERIES

§ 1. **Fourier's Expansion.** In the course of his researches on the Conduction of Heat, Fourier was led (1807-11) to the discovery of the theorem that a function $f(x)$ can usually be expressed in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \qquad . \quad (1)$$

where $-\pi \leqq x \leqq \pi$.

If it is assumed, for the time being, that the series in (1) can be integrated term by term between the limits $-\pi$ and π, the coefficients can be determined as follows. First of all, integrate both sides of the equation (1) over the range $(-\pi, \pi)$; this gives

$$\int_{-\pi}^{\pi} f(x)dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-\pi}^{\pi} \cos nx\, dx + b_n\int_{-\pi}^{\pi} \sin nx\, dx\right\}$$

$$= 2\pi a_0,$$

since $$\int_{-\pi}^{\pi} \cos nx\, dx = 0, \quad \int_{-\pi}^{\pi} \sin nx\, dx = 0.$$

In the next place, multiply (1) by $\cos mx$, where m is a positive integer, and integrate. When m and n are unequal

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \{\cos (n+m)x + \cos (n-m)x\}dx = 0,$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \{\sin (n+m)x + \sin (n-m)x\}dx = 0,$$

so that

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m\int_{-\pi}^{\pi}\cos^2 mx\,dx + b_m\int_{-\pi}^{\pi}\sin mx\cos mx\,dx$$
$$= \tfrac{1}{2}a_m\int_{-\pi}^{\pi}(1+\cos 2mx)dx + \tfrac{1}{2}b_m\int_{-\pi}^{\pi}\sin 2mx\,dx$$
$$= \pi a_m.$$

Again, multiply (1) by sin mx, and integrate ; then, since, when m and n are unequal,

$$\int_{-\pi}^{\pi}\sin nx\sin mx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\{\cos(n-m)x - \cos(n+m)x\}dx = 0,$$
$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_m\int_{-\pi}^{\pi}\cos mx\sin mx\,dx + b_m\int_{-\pi}^{\pi}\sin^2 mx\,dx$$
$$= \tfrac{1}{2}b_m\int_{-\pi}^{\pi}(1-\cos 2mx)\,dx = \pi b_m.$$

Thus

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx,\quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \qquad (2)$$

so that (1) can be written

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos nx\int_{-\pi}^{\pi} f(y)\cos ny\,dy\right.$$
$$\left. + \sin nx\int_{-\pi}^{\pi} f(y)\sin ny\,dy\right\} \qquad (3)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(y)\cos n(y-x)dy \quad . \qquad (4)$$

The series on the right of (3) is a Fourier Series. In determining the coefficients the following assumptions have been made :—

(i) That an expansion for $f(x)$ in a series of sines and cosines of integral multiples of x is possible ;

(ii) that the series can be integrated term by term.

Thus it is still necessary to prove that the series does give the values of $f(x)$ for the range $-\pi \leqq x \leqq \pi$, and to determine what restrictions on $f(x)$ are required in order that the theorem may be valid.

It should be noted that the series is periodic, of period

2π; so that, if it gives the values of $f(x)$ for values of x between $-\pi$ and π, it will not, in general, give the values of $f(x)$ for values of x outside these limits: indeed, it could only do so if $f(x)$ were itself periodic, with a period 2π.

If the function $f(x)$ is even; that is, if, for $-\pi \leqq x \leqq \pi$, $f(-x) = f(x)$, then all the coefficients b_m will vanish. For, from (2),

$$\begin{aligned}\pi b_m &= \int_{-\pi}^{0} f(x)\sin mx\,dx + \int_0^\pi f(x)\sin mx\,dx\\ &= \int_\pi^0 f(-x)\sin mx\,dx + \int_0^\pi f(x)\sin mx\,dx\\ &= -\int_0^\pi f(x)\sin mx\,dx + \int_0^\pi f(x)\sin mx\,dx = 0.\end{aligned}$$

Thus, for an even function, equation (3) becomes

$$f(x) = \frac{1}{\pi}\int_0^\pi f(y)dy + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\int_0^\pi f(y)\cos ny\,dy. \qquad (5)$$

Again, if $f(x)$ is odd; that is, if $f(-x) = -f(x)$ for $-\pi \leqq x \leqq \pi$, then, from (2),

$$\begin{aligned}a_m &= \frac{1}{\pi}\int_{-\pi}^{0} f(y)\cos my\,dy + \frac{1}{\pi}\int_0^\pi f(y)\cos my\,dy\\ &= -\frac{1}{\pi}\int_0^\pi f(y)\cos my\,dy + \frac{1}{\pi}\int_0^\pi f(y)\cos my\,dy = 0,\end{aligned}$$

and similarly $a_0 = 0$, so that every a_m is zero. Hence, for an odd function, (3) becomes

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nx\int_0^\pi f(y)\sin ny\,dy \quad . \quad . \quad (6)$$

The following examples will illustrate the method of evaluating the coefficients in a Fourier Series.

Example 1. Find the Fourier Series for the function $f(x)$, where $f(x) = 0$ for $-\pi \leqq x \leqq 0$, and $f(x) = x$ for $0 \leqq x \leqq \pi$.

From (2)

$$a_0 = \frac{1}{2\pi}\int_0^\pi x\,dx = \tfrac{1}{4}\pi,$$

$$a_m = \frac{1}{\pi}\int_0^\pi x\cos mx\,dx = \frac{1}{\pi}\left[x\frac{\sin mx}{m} + \frac{\cos mx}{m^2}\right]_0^\pi = \begin{cases}0, \text{ if } m \text{ is even,}\\ \dfrac{-2}{\pi m^2}, \text{ if } m \text{ is odd,}\end{cases}$$

$$b_m = \frac{1}{\pi}\int_0^\pi x\sin mx\,dx = \frac{1}{\pi}\left[-x\frac{\cos mx}{m} + \frac{\sin mx}{m^2}\right]_0^\pi = \frac{(-1)^{m-1}}{m}.$$

Hence

$$f(x) = \tfrac{1}{4}\pi - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$
$$+ \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right).$$

When $x = 0$, it follows from the formula

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

that the value of the series is zero, which is also the value of $f(x)$. On the other hand, when $x = \pm\pi$, the series has the value $\tfrac{1}{2}\pi$, while $f(-\pi) = 0$ and $f(\pi) = \pi$; so that the value of the series is the mean of these two values of $f(x)$.

Example 2 (*Even Function*). Let $f(x) = -x$ for $-\pi \leqq x \leqq 0$, and $f(x) = x$ for $0 \leqq x \leqq \pi$; then

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx, \quad a_m = \frac{2}{\pi}\int_0^\pi x \cos mx\,dx,$$

so that these constants have twice the values of the corresponding constants in *ex.* 1, while $b_m = 0$; thus

$$f(x) = \tfrac{1}{2}\pi - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

If $x = 0$, the series and $f(x)$ have both the value zero, while, if $x = \pm\pi$, the series and $f(x)$ are both equal to π.

Example 3 (*Odd Function*). Let the function $f(x)$ be given by

$$f(x) = \begin{cases} -1, & -\pi \leqq x < 0, \\ 0, & x = 0, \\ 1, & 0 < x \leqq \pi. \end{cases}$$

Then $a_m = 0$ and

$$b_m = \frac{2}{\pi}\int_0^\pi \sin mx\,dx = \begin{cases} 0, \text{ if } m \text{ is even,} \\ \dfrac{4}{m\pi}, \text{ if } m \text{ is odd;} \end{cases}$$

thus

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right).$$

When $x = 0$, the series and the function have each the value zero; on the other hand, $f(-\pi) = -1$ and $f(\pi) = 1$, while for $x = \pm\pi$ the series has the value zero, the mean of these two values.

The graph of the sum of n terms of the Fourier Series for a function $f(x)$ approximates to the graph of $f(x)$, the greater the value of n the closer being the approximation. In Figures 1 and 2 are shown the graphs of the functions $y = f(x)$ in *examples*

2 and 3, with, in each case, the graph of the sum of the first three terms of the corresponding Fourier Series.

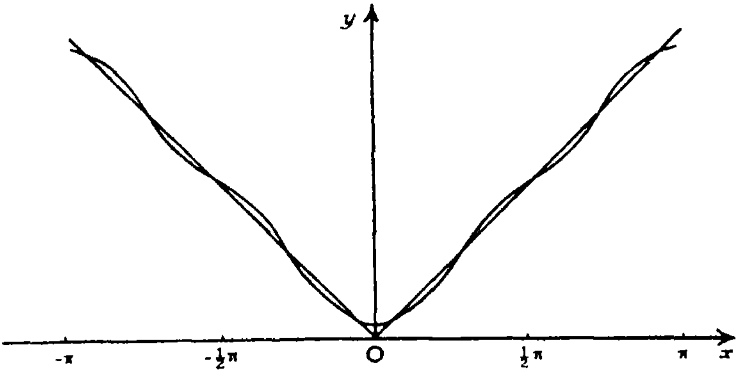

FIG. 1.

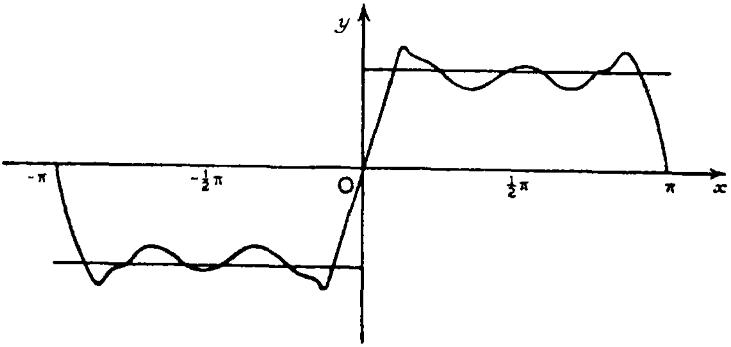

FIG. 2.

§ 2. **Validity of the Expansion.** The simplest method of establishing the validity of the Fourier Expansion is to take the sum of the first $(n + 1)$ terms of the series in (4), and show that, as n tends to infinity, this sum tends to $f(x)$. Let S_{n+1} denote this sum; then

$$S_{n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left\{\frac{1}{2} + \sum_{r=1}^{n} \cos r(y - x)\right\} dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\, \frac{\sin(2n + 1)\dfrac{y - x}{2}}{2\sin\dfrac{y - x}{2}}\, dy.$$

Here put $y - x = 2u$; then

$$
\begin{aligned}
S_{n+1} &= \frac{1}{\pi}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} f(x+2u)\,\frac{\sin(2n+1)u}{\sin u}\,du \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi-x}{2}} f(x+2u)\,\frac{\sin(2n+1)u}{\sin u}\,du \\
&\qquad + \frac{1}{\pi}\int_{-\frac{\pi+x}{2}}^{0} f(x+2u)\,\frac{\sin(2n+1)u}{\sin u}\,du \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi-x}{2}} f(x+2u)\,\frac{\sin(2n+1)u}{\sin u}\,du \\
&\qquad + \frac{1}{\pi}\int_{0}^{\frac{\pi+x}{2}} f(x-2u)\,\frac{\sin(2n+1)u}{\sin u}\,du, \qquad (7)
\end{aligned}
$$

on replacing u by $-u$ in the second integral.

The problem now is to determine the limits to which these integrals tend when n tends to infinity. The integrals can be expressed in the forms known as Dirichlet's Integrals, which will be discussed in the next section. It will be assumed that the function $f(x)$ satisfies the following conditions, known as *Dirichlet's Conditions*:

(i) The function must be continuous at all points of the interval under consideration, except that it may have a finite number of finite discontinuities, like the discontinuity at $x = 0$ in *ex.* 3 of § 1;

(ii) There must only be a finite number of turning points of the function in the interval; an example of such a point is given in *ex.* 2 of § 1 at $x = 0$. The function $\sin\frac{1}{x}$ has an infinite number of maxima and minima near $x = 0$, and does not therefore satisfy the required conditions.

It is possible to extend these conditions to include wider classes of functions, but these extensions are not required for the purposes of the present volume.

Since the number of discontinuities is finite, each discontinuity will be isolated; that is, at a discontinuity $x = c$ a definite positive quantity η can be found such that, in the interval $(c - \eta, c + \eta)$, $f(x)$ is continuous except at $x = c$. The interval $(c - \eta, c + \eta)$ is called *the neighbourhood of* c.

If the function $f(x)$ has an isolated finite discontinuity at $x = c$, $f(x)$ tends to a definite limit as x tends to c through values greater than c, and the value of this limit is denoted by $f(c+0)$: similarly, $f(x)$ tends to a definite limit $f(c-0)$ when x tends to c through values less than c. In Fig. 3, $OM = c$, AP_2 and P_1Q are parts of the curve $y = f(x)$, $MP_1 = f(c+0)$, and $MP_2 = f(c-0)$.

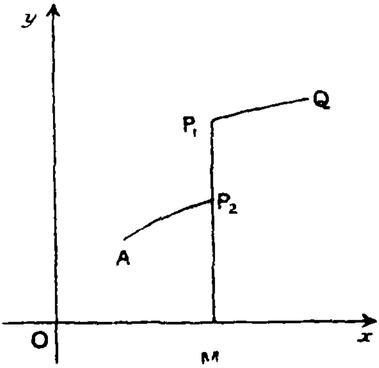

FIG. 3.

§ 3. **Dirichlet's Integrals.** Consider the integral

$$\int_a^b \phi(u) \sin mu \, du, \quad . \quad . \quad . \quad (8)$$

where m is any positive number, and $\phi(u)$ satisfies the Dirichlet's Conditions of the previous article. The limits a and b and the function $\phi(u)$ may involve a parameter: for instance, x is a parameter in the integrals in (7).

Now let $a_1, a_2, \ldots, a_p$ be the values of u in the interval (a, b) for which $\phi(u)$ has either a turning value or a discontinuity; then, in each of the $p+1$ intervals (a, a_1), $(a_1, a_2) \ldots$, (a_p, b), $\phi(u)$ is monotonic; that is, as u increases $\phi(u)$ either never decreases or never increases in the interval. The second theorem of mean value can therefore be applied to the integral (8) in each of these intervals; so that

$$\int_{a_r}^{a_{r+1}} \phi(u) \sin mu \, du$$

$$= \phi(a_r)\int_{a_r}^{\xi} \sin mu \, du + \phi(a_{r+1})\int_{\xi}^{a_{r+1}} \sin mu \, du$$

$$= \phi(a_r)\frac{\cos ma_r - \cos m\xi}{m} + \phi(a_{r+1})\frac{\cos m\xi - \cos ma_{r+1}}{m},$$

where $a_r \leqq \xi \leqq a_{r+1}$; thus

$$\left|\int_{a_r}^{a_{r+1}} \phi(u) \sin mu \, du\right| \leqq \frac{2}{m}\left|\phi(a_r)\right| + \frac{2}{m}\left|\phi(a_{r+1})\right|$$
$$< \frac{4G}{m},$$

where G is a positive number which is greater than $|\phi(u)|$ for all values of u in the interval (a, b). If a parameter x is involved, G is chosen so that the inequality holds for all values of x under consideration.

Hence

$$\left|\int_a^b \phi(u) \sin mu \, du\right| < \frac{4(p+1)G}{m}, \quad . \quad . \quad (9)$$

where, if a parameter x is involved, the greatest value that p can have for any value of x under consideration is taken.

Accordingly, since G and p are finite and independent of x, when m tends to infinity the integral (8) tends uniformly to zero.

In the same way it can be shown that the integral

$$\int_a^b \phi(u) \cos mu \, du \quad . \quad . \quad . \quad (10)$$

tends uniformly to zero when m tends to infinity.

Next, it can be shown that the integral

$$\int_a^b \phi(u) \frac{\sin mu}{u} \, du, \quad . \quad . \quad . \quad (11)$$

where $0 < a < b$, tends uniformly to zero as m tends to infinity. For

$$\int_{a_r}^{a_{r+1}} \phi(u) \frac{\sin mu}{u} \, du = \phi(a_r)\int_{a_r} \frac{\sin mu}{u} \, du$$
$$+ \phi(a_{r+1})\int_{\xi}^{a_{r+1}} \frac{\sin mu}{u} \, du$$
$$= \phi(a_r)\int_{ma_r}^{m\xi} \frac{\sin u}{u} \, du + \phi(a_{r+1})\int_{m\xi}^{ma_{r+1}} \frac{\sin u}{u} \, du;$$

and, since the integral

$$\int_0^{\infty} \frac{\sin u}{u} \, du$$

is convergent, and each of the quantities a_r, ξ, a_{r+1} is not less than the positive quantity a, a definite positive quantity M (independent of x) can be found such that, for $m \geqq M$,

$$\left| \int_{ma_r}^{m\xi} \frac{\sin u}{u}\, du \right| < \epsilon, \quad \left| \int_{m\xi}^{ma_r+1} \frac{\sin u}{u}\, du \right| < \epsilon,$$

where ϵ is any assigned positive quantity. Hence

$$\left| \int_{a_r}^{a_r+1} \phi(u) \frac{\sin mu}{u}\, du \right| < \epsilon \mid \phi(a_r) \mid + \epsilon \mid \phi(a_{r+1}) \mid$$
$$< 2\epsilon G,$$

and
$$\left| \int_a^b \phi(u) \frac{\sin mu}{u}\, du \right| < 2(p+1)\epsilon G.$$

Thus the integral (11) converges uniformly to zero as m tends to infinity.

Lastly, it can be shown that the integral

$$\int_0^a \phi(u) \frac{\sin mu}{u}\, du, \qquad . \quad . \quad . \quad (12)$$

where a is positive, tends to the value $\frac{1}{2}\pi\phi(+0)$ when m tends to infinity; if $\phi(u)$ is continuous at $u = 0$, this limit is $\frac{1}{2}\pi\phi(0)$.

Let k be a positive number such that $0 < k < a$; then,

$$\int_0^a \phi(u) \frac{\sin mu}{u}\, du = \int_0^k \phi(u) \frac{\sin mu}{u}\, du + \int_k^a \phi(u) \frac{\sin mu}{u}\, du.$$

Since the second integral is of the type (11) it tends uniformly to zero as m tends to infinity. Now, choose k so small that $\phi(u)$ is monotonic in the interval $(0, k)$; then, by the second theorem of mean value

$$\int_0^k \phi(u) \frac{\sin mu}{u}\, du = \phi(+0) \int_0^k \frac{\sin mu}{u}\, du$$
$$+ \{\phi(k) - \phi(+0)\} \int_\xi^k \frac{\sin mu}{u}\, du,$$

where $0 \leqq \xi \leqq k$: thus

$$\int_0^k \phi(u) \frac{\sin mu}{u}\, du = \phi(+0) \int_0^{mk} \frac{\sin u}{u}\, du$$
$$+ \{\phi(k) - \phi(+0)\} \int_{m\xi}^{mk} \frac{\sin u}{u}\, du \quad (13)$$

When m tends to infinity, the first integral on the right of this equation tends to $\frac{1}{2}\pi$, while the second is numerically not greater than[*] π. But k can be chosen so small that, for any assigned positive quantity ϵ,

$$|\phi(k) - \phi(+0)| < \epsilon \quad . \quad . \quad . \quad (14)$$

Therefore, when m tends to infinity, the integral (12) tends to the limit $\frac{1}{2}\pi\phi(+0)$.

When there is a parameter x, it may happen that it is not possible to determine a fixed k so that inequality (14) should hold for all values of x under consideration; in that case, while the first integral can always be made to approach indefinitely near $\frac{1}{2}\pi$, it may not do so uniformly. This point will be dealt with more fully in the following section.

§ 4. **Summation of the Fourier Series.** Returning now to the summation (7), let us assume, to begin with, that the function $f(x)$ is continuous at all points of the interval (p, q), where $-\pi < p < q < \pi$. The first integral in (7) can be written in the form

$$\int_0^{\frac{\pi-x}{2}} f(x+2u)\frac{u}{\sin u}\frac{\sin(2n+1)u}{u}du \quad . \quad . \quad (15)$$

which is of the form (12) with $m = 2n + 1$ and

$$\phi(u) = f(x+2u)\frac{u}{\sin u}.$$

Then, for any assigned ϵ, a k can be found such that the inequality (14) holds for all points x of the interval (p, q). It is then possible to choose a positive quantity M so that, when $m \geqq \mathrm{M}$, the first integral in (13) differs from $\frac{1}{2}\pi$ by less than any assigned small positive quantity. Thus the integral on the left of (13) tends uniformly to $\frac{1}{2}\pi\phi(+0)$ when m tends to infinity; therefore (15) tends uniformly to $\frac{1}{2}\pi f(x+0)$ when m tends to infinity.

In the same way it can be shown that the second integral in (7) tends uniformly to $\frac{1}{2}\pi f(x-0)$ for all values of x in the interval (p, q). Thus S_{n+1} tends uniformly to

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[*] The reader may verify this statement by drawing the curve $y = \sin x/x$: the successive waves have the same breadth π and diminish in amplitude, the greatest ordinate being $y = 1$ when $x = 0$: thus if $u_1, -u_2, u_3, -u_4 \ldots$ be the areas between the ordinates at $x = 0, \pi, 2\pi, 3\pi, \ldots, \pi > u_1 > u_2 > u_3 \ldots$

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

and this is equal to $f(x)$, since the function is continuous at the point x.

Point of Discontinuity. At a point of discontinuity of $f(x)$ it is, of course, impossible to include x in an interval of continuity (p, q), and, in consequence, the convergence to the limit of one of the integrals in (7) ceases to be uniform. From equation (13), however, it follows that, though the convergence is not uniform,

$$\operatorname*{Lim}_{n \to \infty} S_{n+1} = \tfrac{1}{2}\{f(x+0)+f(x-0)\} \quad . \quad (16)$$

An illustration of a discontinuity of this type will be found in *ex.* 3 of § 1.

*Note 1. These results are in accordance with the well-*known theorem that the sum of a uniformly convergent series, the terms of which are continuous functions of the variable, is itself a continuous function.

When $x = \pi$, (7) becomes

$$S_{n+1} = \frac{1}{\pi}\int_0^{\pi} f(\pi - 2u)\,\frac{\sin(2n+1)u}{\sin u}\,du$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} f(\pi - 2u)\,\frac{\sin(2n+1)u}{\sin u}\,du$$

$$+ \frac{1}{\pi}\int_0^{\frac{\pi}{2}} f(-\pi + 2u)\,\frac{\sin(2n+1)u}{\sin u}\,du,$$

as can be seen by dividing the range $(0, \pi)$ into the two parts $(0, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \pi)$, and in the second substituting $\pi - u$ for u in the integral: hence, as before

$$\operatorname*{Lim}_{n \to \infty} S_{n+1} = \tfrac{1}{2}\{f(\pi-0)+f(-\pi+0)\} \quad . \quad (17)$$

If $f(-\pi) = f(\pi)$, the sum of the series is $f(\pm\pi)$. For the interval $(\pi\ 3\pi)$ the function $f(x)$ can be defined by means of the equation $f(x) = f(-2\pi + x)$: the point π can then be included in an interval of continuity of the function, and, consequently, the convergence is uniform. When $f(\pi)$ and $f(-\pi)$ are unequal, the points π and $-\pi$ are to be counted among the points of discontinuity.

The value of the series when $x = -\pi$ is also given by (17).

Note 2. From the formulæ (2) and (9) it follows that the coefficients a_n, b_n in the series (1) are less in absolute magnitude than $\frac{C}{n}$, where C is a definite constant; thus, while the series is convergent, it is not usually absolutely convergent.

Note 3. If the function $f(x)$ is continuous, and has equal values when $x = \pm\pi$, and if it has a derivative $f'(x)$ which satisfies Dirichlet's Conditions (§ 2), then, from (2), by partial integration

$$\begin{aligned} b_n &= -\frac{1}{\pi}\left[f(x)\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx \\ &= \frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx, \end{aligned}$$

and similarly

$$a_n = -\frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx,$$

so that, from (9),

$$|a_n| < \frac{C}{n^2} \quad \text{and} \quad |b_n| < \frac{C}{n^2}.$$

In this case the series will be absolutely and uniformly convergent, and $f'(x)$ can be obtained by differentiating the series for $f(x)$.

§ 5. **Sine Series and Cosine Series.** The validity of the series (5) and (6) for the range $-\pi \leqq x \leqq \pi$ follows from that of (4) for the cases $f(x)$ even and $f(x)$ odd respectively, since (5) and (6) are then merely particular cases of (4). It can easily be shown, however, that (5) and (6) still hold for the more restricted range $0 \leqq x \leqq \pi$, when $f(x)$ is any function satisfying Dirichlet's Conditions.

For instance, if $f(x)$ is not an even function, let a function $F(x)$ be defined as follows:

$$\begin{aligned} F(x) &= f(x), & 0 \leqq x \leqq \pi, \\ F(x) &= f(-x), & -\pi \leqq x \leqq 0. \end{aligned}$$

Then $F(x)$ is an even function, and, for the range $-\pi \leqq x \leqq \pi$, its value is given by (5). But, for the range $0 \leqq x \leqq \pi$, $F(x) = f(x)$; hence, for this range, $f(x)$ is given by (5).

Note. In this case $F(x)$ is continuous at $x = 0$, and therefore the actual value of $f(0)$ is given by the series. Also, $F(-\pi) = F(\pi)$, so that the value of $f(\pi)$ is given by the series.

Again, when $f(x)$ is not an odd function, let $F(x)$ be defined by the equations

$$\begin{aligned} F(x) &= f(x), & 0 \leqq x \leqq \pi, \\ F(x) &= -f(-x), & -\pi \leqq x \leqq 0, \end{aligned}$$

Then $F(x)$ is an odd function, for which the expansion (6) is valid, and this expansion holds for $f(x)$ for the range $0 < x < \pi$. When $x = 0$ and when $x = \pi$ the series is zero, so that it will not represent the function at these points unless $f(0)$ and $f(\pi)$ are zero.

Note. For the cosine series the equivalent of the conditions of *Note* 3 of last section is that, in the range $0 \leqq x \leqq \pi$, $f(x)$ must be continuous, and $f'(x)$ must satisfy Dirichlet's Conditions. For the sine series, in addition, $f(x)$ must vanish when $x = 0$ and when $x = \pi$.

§ 6. **Other Forms of Fourier Series.** Fourier expansions are sometimes required for other ranges of x than those discussed above. For example, the expansion for the range $0 \leqq x \leqq 2\pi$ can be deduced from that of (3) by writing

$$F(y) = f(x),$$

where $y = x - \pi$; then, for $-\pi \leqq y \leqq \pi$,

$$F(y) = a_0 + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny),$$

where, by (2),

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(y)dy = \frac{1}{2\pi}\int_0^{2\pi} F(x-\pi)dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(y)dy,$$

and, similarly,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos n(x - \pi)\, dx = \frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin n(x - \pi)\, dx = \frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx.$$

Hence, for $0 \leqq x \leqq 2\pi$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n(x - \pi) + b_n \sin n(x - \pi)\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos nx \int_0^{2\pi} f(y) \cos ny\, dy \right.$$

$$\left. + \sin nx \int_0^{2\pi} f(y) \sin ny\, dy\right\} \quad (18)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(y) \cos n(y - x)\, dy \quad . \quad (19)$$

Again, for the range $-l \leqq x \leqq l$, let $F(y) = f(x)$, where $y = \pi x/l$; then, if $-\pi \leqq y \leqq \pi$,

$$F(y) = a_0 + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny),$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(y)dy = \frac{1}{2l}\int_{-l}^{l} f(x)dx,$$

and, similarly,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx,\ b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx \quad (20)$$

Hence, with these values of the coefficients,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2l}\int_{-l}^{l} f(y)dy + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(y) \cos\frac{n\pi(y - x)}{l}\, dy \quad (21)$$

where $-l \leqq x \leqq l$.

The reader should notice that in these expansions the coefficient can be at once determined by multiplying by the cosine or sine in the term and integrating over the range under consideration.

As in the previous section it can be shown that the cosine series and the sine series corresponding to (21) are

$$f(x) = \frac{1}{l}\int_0^{l} f(y)dy + \frac{2}{l}\sum_{n=1}^{\infty} \cos\frac{n\pi x}{l}\int_0^{l} f(y) \cos\frac{n\pi y}{l}\, dy, \quad (22)$$

where $\quad 0 \leqq x \leqq l,$

and
$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy, \quad . \qquad (23)$$

where $\quad 0 \leqq x \leqq l.$

Also, from (18), for the range $0 \leqq x \leqq 2l$, it can be deduced that

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right),$$

where
$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\,;$$

or

$$f(x) = \frac{1}{2l}\int_0^{2l} f(y)dy + \frac{1}{l}\sum_{n=1}^{\infty}\int_0^{2l} f(y)\cos\frac{n\pi(y-x)}{l}\,dy. \qquad (24)$$

The reader can easily verify that, for any range $a \leqq x \leqq b$, the expansion becomes

$$f(x) = \frac{1}{b-a}\int_a^b f(y)dy + \frac{2}{b-a}\sum_{n=1}^{\infty}\int_a^b f(y)\cos\frac{2n\pi(y-x)}{b-a}\,dy, \qquad (25)$$

while the cosine and sine series for this range are

$$f(x) = \frac{1}{b-a}\int_a^b f(y)dy + \frac{2}{b-a}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{b-a}\right)\int_a^b f(y)\cos\left(\frac{n\pi y}{b-a}\right)dy, \qquad (26)$$

and

$$f(x) = \frac{2}{b-a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{b-a}\right)\int_a^b f(y)\sin\left(\frac{n\pi y}{b-a}\right)dy. \qquad (27)$$

***Example* 1.** Show that the cosine and sine series for $f(x) = x$ in the interval $0 \leqq x \leqq \pi$ are

(i) $x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right),\ 0 \leqq x \geqq \pi,$

(ii) $x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right),\ 0 \leqq x < \pi.$

Since the first expansion holds when $x = 0$, it can be employed to obtain the formula

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots .$$

Example 2. If m is neither zero nor an integer, show that

$$\text{(i)}\ \sin mx = \frac{2}{\pi} \sin m\pi \left(\frac{\sin x}{1^2 - m^2} - \frac{2 \sin 2x}{2^2 - m^2} + \frac{3 \sin 3x}{3^2 - m^2} - \ldots \right),$$

where $-\pi < x < \pi$;

$$\text{(ii)}\ \cos mx = \frac{2}{\pi} \sin m\pi \left(\frac{1}{2m} + \frac{m \cos x}{1^2 - m^2} - \frac{m \cos 2x}{2^2 - m^2} + \frac{m \cos 3x}{3^2 - m^2} - \ldots \right),$$

where $-\pi \leqq x \leqq \pi$.

From (ii), by putting $x = 0$ and $x = \pi$, deduce that

$$\text{(iii)}\ \frac{\pi}{\sin m\pi} = \frac{1}{m} - \frac{2m}{m^2 - 1^2} + \frac{2m}{m^2 - 2^2} - \frac{2m}{m^2 - 3^2} + \ldots,$$

$$\text{(iv)}\ \pi \cot m\pi = \frac{1}{m} + \frac{2m}{m^2 - 1^2} + \frac{2m}{m^2 - 2^2} + \ldots .$$

Example 3. Show that, if m is not zero, and $-\pi < x < \pi$,

$$\cosh mx = \frac{\sinh m\pi}{m\pi} + \frac{2m \sinh m\pi}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{m^2 + n^2}.$$

Deduce that

$$\text{(i)}\ \frac{\pi}{\sinh m\pi} = \frac{1}{m} + \sum_{n=1}^{\infty} (-1)^n \frac{2m}{m^2 + n^2},$$

$$\text{(ii)}\ \pi \coth m\pi = \frac{1}{m} + \sum_{n=1}^{\infty} \frac{2m}{m^2 + n^2}.$$

§ 7. **Fourier's Double Integral.** From the integrals (11) and (12) it follows that, when m tends to infinity, the integral

$$\int_a^b \phi(u) \frac{\sin mu}{u}\, du \quad . \quad . \quad . \quad (28)$$

tends to zero if $0 < a < b$, and to $\frac{1}{2}\pi\phi(+0)$ if $0 = a < b$, provided that $\phi(u)$ satisfies Dirichlet's Conditions (§ 2). Also, if $a < b < 0$,

$$\int_a^b \phi(u) \frac{\sin mu}{u}\, du = \int_{-b}^{-a} \phi(-u) \frac{\sin mu}{u}\, du \quad . \quad (29)$$

and, from (28), when m tends to infinity this tends to zero, while, if $a < b = 0$, it tends to $\frac{1}{2}\pi\phi(-0)$. Also, if $a < 0 < b$,

$$\int_a^b \phi(u)\frac{\sin mu}{u}\,du = \int_a^0 \phi(u)\frac{\sin mu}{u}\,du + \int_0^b \phi(u)\frac{\sin mu}{u}\,du,$$

and, from (28) and (29), when m tends to infinity this tends to

$$\tfrac{1}{2}\pi\{\phi(+0) + \phi(-0)\}.$$

Here replace $\phi(u)$ by $f(x+u)$, and these results can be written

$$\operatorname*{Lim}_{m\to\infty} \frac{1}{\pi}\int_a^b f(x+u)\frac{\sin mu}{u}\,du = \begin{cases} \frac{1}{2}\{f(x+0)+f(x-0)\}, & \text{if } a<0<b, \\ \frac{1}{2}f(x+0), & \text{if } 0=a<b, \\ \frac{1}{2}f(x-0), & \text{if } a<b=0, \\ 0, & \text{if } 0<a<b \text{ or } a<b<0. \end{cases} \qquad (30)$$

Now let it be assumed that the integral

$$\int_{-\infty}^{\infty} f(x+u)\frac{\sin mu}{u}\,du \quad . \quad . \quad . \quad (31)$$

converges uniformly for all values of m; that is, that corresponding to any positive quantity ϵ, however small, a positive quantity K can be found such that, for all values of m,

$$\left|\int_k^\infty f(x+u)\frac{\sin mu}{u}\,du\right| < \epsilon, \text{ and } \left|\int_{-\infty}^{-k} f(x+u)\frac{\sin mu}{u}\,du\right| < \epsilon,$$

provided that $k > \text{K}$: then, from (30),

$$\operatorname*{Lim}_{m\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin mu}{u}\,du = \tfrac{1}{2}\{f(x+0) + f(x-0)\}. \quad (32)$$

In (32) write α for $x + u$, and it becomes

$$\operatorname*{Lim}_{m\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(\alpha)\frac{\sin m(\alpha-x)}{\alpha - x}\,d\alpha = \tfrac{1}{2}\{f(x+0)+f(x-0)\}. \quad (33)$$

Now

$$\frac{\sin m(\alpha - x)}{\alpha - x} = \int_0^m \cos\beta(\alpha - x)\,d\beta;$$

so that, from (33),

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \operatorname*{Lim}_{m\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha\int_0^m \cos\beta(\alpha - x)\,d\beta \qquad (34)$$

$$= \operatorname*{Lim}_{m\to\infty} \frac{1}{\pi}\int_0^m d\beta\int_{-\infty}^{\infty} f(\alpha)\cos\beta(\alpha - x)\,d\alpha \qquad (35)$$

$$= \frac{1}{\pi}\int_0^\infty d\beta\int_{-\infty}^{\infty} f(\alpha)\cos\beta(\alpha - x)\,d\alpha, \quad . \quad (36)$$

provided that it is allowable to change the order of integration. The integral (36) is known as *Fourier's Double Integral.* Of course, when $f(x)$ is continuous, the value of the integral is $f(x)$.

When $f(x)$ is even, the second integral in (36) may be written

$$\int_0^\infty f(\alpha)\cos\beta(\alpha - x)\,d\alpha + \int_0^\infty f(\alpha)\cos\beta(\alpha + x)\,d\alpha$$
$$= 2\int_0^\infty f(\alpha)\cos\alpha\beta\cos x\beta\,d\alpha\,;$$

so that (36) becomes

$$\frac{2}{\pi}\int_0^\infty \cos x\beta\,d\beta\int_0^\infty f(\alpha)\cos\alpha\beta\,d\alpha = \tfrac{1}{2}\{f(x+0)+f(x-0)\}. \quad (37)$$

Similarly, if $f(x)$ is odd, (36) becomes

$$\frac{2}{\pi}\int_0^\infty \sin x\beta\,d\beta\int_0^\infty f(\alpha)\sin\alpha\beta\,d\alpha = \tfrac{1}{2}\{f(x+0)+f(x-0)\}. \quad (38)$$

For positive values of x the formula (37) is still valid, even when $f(x)$ is not even. This can be deduced from (36) by substituting $f(-x)$ for $f(x)$ in that formula when x is negative. Similarly (38) still holds for positive values of x even when $f(x)$ is not odd; to prove this, substitute $-f(-x)$ for $f(x)$ in (36) when x is negative.

If in (37) we replace the inner integral by $\sqrt{\left(\frac{\pi}{2}\right)}\phi(\beta)$ we obtain the theorem that, if x is positive and if

$$\int_0^\infty f(\alpha)\cos x\alpha\,d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}\phi(x),$$

then
$$\int_0^\infty \phi(\alpha)\cos x\alpha\,d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}f(x) \quad . \quad . \quad (39)$$

Similarly, from (38) it follows that, if x is positive and if

$$\int_0^\infty f(\alpha)\sin x\alpha\,d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}\phi(x),$$

then
$$\int_0^\infty \phi(\alpha)\sin x\alpha\,d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}f(x) \quad . \quad . \quad (40)$$

It will now be shown that, if the integral

$$\int_{-\infty}^{\infty} f(x)dx$$

is absolutely convergent, all the steps in the proof just given are valid. For then a positive quantity K can be found such that, if $k \geqq \mathrm{K}$,

$$\left| \int_k^{\infty} f(x+u) \frac{\sin mu}{u}\, du \right| \leqq \int_k^{\infty} \frac{|f(x+u)|}{u} du$$

$$< \frac{1}{k}\int_k^{\infty} |f(x+u)| du < \frac{\epsilon}{k}, \qquad (40)$$

and similarly

$$\left| \int_{-\infty}^{-k} f(x+u) \frac{\sin mu}{u}\, du \right| < \frac{\epsilon}{k},$$

for all values of m. Thus the integral (31) converges uniformly, and consequently (32), (33), and (34) are valid.

Now

$$\int_0^{\infty} f(\alpha)d\alpha \int_0^m \cos\beta(\alpha - x)\, d\beta - \int_0^m d\beta \int_0^{\infty} f(\alpha)\cos\beta(\alpha - x)\, d\alpha$$

$$= \int_0^k f(\alpha)d\alpha \int_0^m \cos\beta(\alpha - x)\, d\beta - \int_0^m d\beta \int_0^k f(\alpha)\cos\beta(\alpha - x)\, d\alpha$$

$$+ \int_k^{\infty} f(\alpha)d\alpha \int_0^m \cos\beta(\alpha - x)\, d\beta - \int_0^m d\beta \int_k^{\infty} f(\alpha)\cos\beta(\alpha - x)\, d\alpha,$$

and, no matter how large m and k are, the first two integrals cancel. From (40), the absolute value of the third integral is less than ϵ/k; also, for the fourth integral,

$$\left| \int_0^m d\beta \int_k^{\infty} f(\alpha)\cos\beta(\alpha - x)\, d\alpha \right| \leqq \int_0^m d\beta \int_k^{\infty} |f(\alpha)|\, d\alpha$$

$$< \epsilon,$$

if K is chosen so large that

$$\int_k^{\infty} |f(\alpha)|\, d\alpha < \frac{\epsilon}{m},$$

for $k \geqq \mathrm{K}$. Thus, no matter how large m is, K can be chosen so large that

$$\left| \int_0^{\infty} f(\alpha)d\alpha \int_0^m \cos\beta(\alpha - x)\, d\beta - \int_0^m d\beta \int_0^{\infty} f(\alpha)\cos\beta(\alpha - x)\, d\alpha \right|$$

$$< \epsilon\left(\frac{1}{k} + 1\right) < 2\epsilon,$$

and therefore, since the integrals are independent of k,

$$\int_0^{\infty} f(\alpha)d\alpha \int_0^m \cos\beta(\alpha - x)\, d\beta = \int_0^m d\beta \int_0^{\infty} f(\alpha)\cos\beta(\alpha - x)\, d\alpha.$$

Similarly

$$\int_{-\infty}^{0} f(\alpha)d\alpha \int_{0}^{m} \cos\beta(\alpha - x)\, d\beta = \int_{0}^{m} d\beta \int_{-\infty}^{0} f(\alpha) \cos\beta(\alpha - x)\, d\alpha\,;$$

and therefore (35) and consequently (36) are valid.

Examples.

1. If $-\pi < x < \pi$, show that

$$e^x = \frac{2\sinh\pi}{\pi}\left(\tfrac{1}{2} - \frac{\cos x}{1^2+1} + \frac{\cos 2x}{2^2+1} - \frac{\cos 3x}{3^2+1} + \dots \right.$$
$$\left. + \frac{1\,.\,\sin x}{1^2+1} - \frac{2\sin 2x}{2^2+1} + \frac{3\sin 3x}{3^2+1} - \dots \right).$$

2. If $f(x) = \pi + x$ for $-\pi \leq x \leq 0$, and $f(x) = \pi - x$ for $0 \leq x \leq \pi$, show that, for $-\pi \leq x \leq \pi$,

$$f(x) = \tfrac{1}{2}\pi + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right).$$

3. Prove that

$$\text{(i)}\ \pi - x = 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right),\ 0 < x \leq \pi,$$

$$\text{(ii)}\ \pi + x = 6\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$$
$$- 2\left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \dots \right),\ 0 < x < \pi.$$

4. Show that, if $0 \leq x \leq \pi$,

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \dots \right).$$

5. Prove that, if $0 < x < \pi$,

$$\cos x = \frac{4}{\pi}\left(\frac{2\sin 2x}{1\cdot 3} + \frac{4\sin 4x}{3\cdot 5} + \frac{6\sin 6x}{5\cdot 7} + \dots \right).$$

6. If $f(x) = 0$ for $0 < x < \tfrac{3}{2}\pi$, and $f(x) = 1$ for $\tfrac{3}{2}\pi < x < 3\pi$, show that, for $0 < x < 3\pi$,

$$f(x) = \tfrac{1}{2} - \frac{2}{\pi}\left(\sin\frac{2x}{3} + \frac{1}{3}\sin\frac{6x}{3} + \frac{1}{5}\sin\frac{10x}{3} + \dots \right).$$

7. If $0 \leqq x \leqq \pi$, show that

$$\text{(i)}\ x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \dots \right),$$

$$\text{(ii)}\ x(\pi - x) = \frac{\pi^2}{6} - \frac{\cos 2x}{1^2} - \frac{\cos 4x}{2^2} - \frac{\cos 6x}{3^2} - \dots .$$

8. If $0 \leqq x \leqq \pi$, show that

$$\text{(i)}\ \tfrac{1}{2}(\pi - x)\sin x = \tfrac{1}{2} + \tfrac{1}{4}\cos x - \frac{1}{1\cdot 3}\cos 2x - \frac{1}{2\cdot 4}\cos 3x$$
$$- \frac{1}{3\cdot 5}\cos 4x - \dots ,$$

(ii) $\frac{1}{2}(\pi - x)\sin x = \frac{1}{4}\pi \sin x + \frac{4}{\pi}\left\{\frac{2\sin 2x}{1^2 . 3^2} + \frac{4\sin 4x}{3^2 . 5^2} + \frac{6 \sin 6x}{5^2 . 7^2} + \dots\right\}$.

9. If $f(x) = \sin x$ for $0 \leqq x < \pi/2$ and $f(x) = -\sin x$ for $\pi/2 < x < \pi$, show that

$$f(x) = \frac{4}{\pi}\left(\frac{2\sin 2x}{1 \cdot 3} - \frac{4 \sin 4x}{3 \cdot 5} + \frac{6 \sin 6x}{5 \cdot 7} - \dots\right).$$

10. If $f(x) = \frac{\sinh \lambda x}{\cosh \lambda a}$ for $0 \leqq x \leqq a$, and $f(x) = \frac{\sinh \lambda(2a - x)}{\cosh \lambda a}$ for $a \leqq x \leqq 2a$, prove that

$$f(x) = \frac{2\lambda}{a}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2a}}{\lambda^2 + \left(\frac{n\pi}{2a}\right)^2}.$$

11. If $f(x) = \sum_{n=-\infty}^{\infty} e^{-\frac{(x+2nl)^2}{c}}$, where c is positive, and if

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l},$$

show that

$$A_n = \frac{2}{l}\int_0^{\infty} e^{-\frac{x^2}{c}} \cos\frac{n\pi x}{l}\,dx.$$

Deduce that

$$f(x) = \frac{\sqrt{(\pi c)}}{2l}\left\{1 + 2\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 c}{4l^2}} \cos\frac{n\pi x}{l}\right\}.$$

12. Show that, when n tends to infinity, the integral

$$\sqrt{n}\int_0^{\frac{\pi}{4}} \frac{\sin nx}{\cos x}\frac{dx}{\sqrt{x}}$$

tends to $\sqrt{(\pi/2)}$.

[Use the Fresnel Integral $\int_0^{\infty} \sin x^2\,dx = \frac{1}{2}\sqrt{(\pi/2)}$.]

13. By integrating each side of the equation

$$\sum_{r=1}^{n} \sin 2rx = \frac{\sin nx \sin(n+1)x}{\sin x}$$

and taking the limit when n tends to infinity, show that, if $0 < \alpha < \beta < \pi$,

$$\sum_{r=1}^{\infty}\frac{\cos 2r\alpha - \cos 2r\beta}{r} = \log \sin\beta - \log \sin\alpha,$$

and that, if $0 < x < \pi$,

$$\sum_{r=1}^{\infty} \frac{\cos rx}{r} = -\log\left(2\sin\frac{x}{2}\right).$$

14. By writing $f(x) = e^{-x}\cos x$ in Fourier's Double Integral, show that

$$\int_0^\infty \frac{x^3 \sin ax\, dx}{x^4 + 4} = \frac{\pi}{2} e^{-a}\cos a,$$

where $a > 0$.

15. If $$\int_0^\infty f(x)\sin\lambda x\, dx = e^{-\lambda^2}\sin\lambda,$$

show that $$f(x) = \frac{1}{2\sqrt{\pi}}\left\{e^{-\frac{(1-x)^2}{4}} - e^{-\frac{(1+x)^2}{4}}\right\}.$$

16. If $$\int_0^\infty f(x)\cos\alpha x\, dx = \frac{\sin\alpha}{\alpha},\text{ find } f(x).$$

[Ans. : $f(x) = 1$ if $0 < x < 1$; $f(x) = 0$ if $x > 1$.]

17. If $\int_0^\infty f(\alpha)\sin x\alpha\, d\alpha = \frac{\sin x}{1 + x^2}$, show that the function $f(x)$ is given by

$$f(x) = \tfrac{1}{2}(e^{-1+x} - e^{-1-x}),\text{ if } 0 \leqq x \leqq 1,$$
$$f(x) = \tfrac{1}{2}(e^{-x+1} - e^{-x-1}),\text{ if } 1 \leqq x.$$

18. Prove that the integral

$$\int_0^\infty \frac{(\sin u - u\cos u)\sin u \sin xu}{u^3}\, du$$

is equal to $\frac{\pi}{8}(2x - x^2)$ if $0 \leqq x \leqq 2$ and is zero if $x > 2$.

19. Show that, if $0 < p < 1$, the integral

$$n^p \int_0^a \cos cx \frac{\sin nx}{x^{1-p}}\, dx$$

tends to $\Gamma(p)\sin\frac{p\pi}{2}$ as n tends to infinity.

CHAPTER II

CONDUCTION OF HEAT

§ 1. **Definitions.** If one part of a solid body is at a higher temperature than another part, heat passes through the body from the hotter to the colder part, and this process is known as *Conduction of Heat.* Conduction should be distinguished from Convection and Radiation. Convection takes place in liquids and gases, and the transference of heat is due to the movements of the particles of the fluid among themselves, the hotter and colder particles mingling with one another. In Radiation heat is transferred from the hotter to the colder body by a process which takes place in an intervening medium; it is thus, for instance, that the earth receives its warmth from the sun.

Conductivity. Consider a slab of solid material bounded by two parallel planes, and let one plane be maintained at a constant temperature v, and the other at a lower constant temperature v_0. It is assumed that the faces of the slab are so large compared with its thickness that, for practical purposes, they may be regarded as infinite in extent. Heat passes from the first to the second face, and after an interval of time the temperature at all points of the slab ceases to vary with the time. If Q denote the amount of heat which then passes in time t across an area S of a section of the slab parallel to the faces, it is found, as a result of experiment, that

$$Q = \frac{K(v - v_0)St}{d}, \quad . \quad . \quad . \quad (1)$$

where d is the thickness of the slab and K is a constant depending on the material of which the slab is formed. This constant is known as the *Thermal Conductivity* of the substance. As a matter of fact, the value of K varies slightly with the temperature, but at ordinary temperatures this variation is so minute that it may be neglected in the Mathematical Theory of Conduction.

In some bodies (heterogeneous bodies) the value of K varies from point to point, but it will here be assumed that the bodies dealt with are homogeneous, so that K is constant throughout. It will also be assumed that the bodies are isotropic, that is, that heat can spread with equal ease in all directions; in crystalline and non-isotropic bodies the rate of conduction may vary with the direction. On the assumption, then, that the bodies dealt with are homogeneous and isotropic, the mathematical theory will be developed from the fundamental hypothesis made in equation (1).

The numerical value of K will depend on the units of measurement adopted, and it will also depend on the definition of the unit of heat employed. The unit of heat will here be taken as that quantity which will raise unit mass of water through one degree centigrade. The dimensions of $Q/(v - v_0)$ will then be simply [M], and consequently the dimensions of K are

$$[K] = \frac{[M]}{[L][T]}.$$

In the C.G.S. system of units the unit of heat is the *Calory*, the quantity of heat which will raise one gramme of water through one degree centigrade.

Specific Heat. It will also be found necessary to make use of the expression "Specific Heat." The specific heat of a body is defined as the amount of heat required to raise unit mass of the body through one degree centigrade. It will be assumed for the purposes of the present volume that the specific heat is independent of the temperature, volume, and pressure.

Isothermal Surfaces. The temperature v at any point (x, y, z) of the body under consideration is a function of the co-ordinates of the point and also of the time t. A surface described in the body so that every point on it at the instant t has the same temperature is called an Isothermal Surface. Obviously, no two isothermal surfaces can intersect, as a point cannot have two temperatures at once; thus the solid may be regarded as divided up into thin shells by its isothermals. At the time t heat is flowing from one isothermal to another, the direction of flow being along the normals to the surfaces, since no transference of heat can take place along surfaces of equal temperature.

Let the temperature of the isothermal through P (x, y, z) at time t be v, $v + \delta v$ the temperature of a neighbouring isothermal, and δn the distance between the two surfaces measured along the normal at P to the surface through P. If the thickness δn of the shell is very small, in the neighbourhood of P the shell may be regarded as a thin slab; and therefore, from (1), the quantity of heat which passes through the isothermal at P per unit of area per unit of time in the direction of the normal n is

$$\frac{K\{v - (v + \delta v)\}}{\delta n} = - K\frac{\delta v}{\delta n};$$

which, when δn tends to zero, tends to

$$- K\frac{\partial v}{\partial n}. \quad . \quad . \quad . \quad . \quad (2)$$

This may be taken, in place of (1), as the fundamental hypothesis in the Mathematical Theory of the Conduction of Heat.

§ 2. **Flow of Heat across any Surface.** It will now be shown that the formula (2) gives the rate of flow of heat per unit area, at time t, not only across an isothermal surface, but across any surface in the solid.

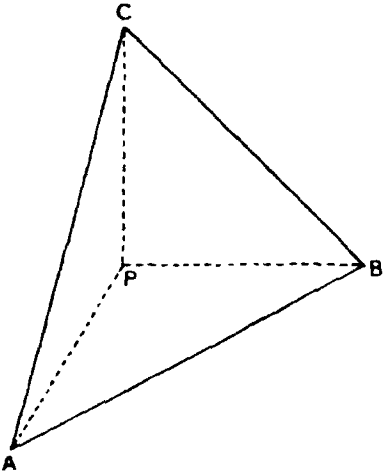

FIG. 4.

Let P be the point (x, y, z) in the solid, and consider a small tetrahedron PABC, in which the faces PBC, PCA, PAB are parallel to the planes $x = 0$, $y = 0$, $z = 0$ respectively. Denote the area of the triangle ABC by Δ, and let p be the length and (λ, μ, ν) the direction-cosines of the perpendicular from P to ABC; the volume of the tetrahedron is then $\frac{1}{3}p\Delta$.

Let f be the rate of flow of heat per unit area outwards across ABC, and let f_x, f_y, f_z be the rates of flow of heat inwards per unit area across PBC, PCA, PAB respectively. Then, since the areas of these triangles are $\lambda\Delta$, $\mu\Delta$, $\nu\Delta$ respectively, the rate at which the tetrahedron is gaining heat is

$$f_x\lambda\Delta + f_y\mu\Delta + f_z\nu\Delta - f\Delta.$$

But if c is the specific heat and ρ the density of the material, this is equal to

$$c\rho\frac{\partial v}{\partial t} \times \tfrac{1}{3}p\Delta,$$

so that

$$\lambda f_x + \mu f_y + \nu f_z - f = \tfrac{1}{3}pc\rho\frac{\partial v}{\partial t} \quad . \quad . \quad (3)$$

Now let the volume, and consequently p tend to zero; then f becomes the rate of flow per unit area at P across a surface whose normal n has direction-cosines (λ, μ, ν), and is called the ***flux*** at P in the direction n. Similarly, f_x, f_y, f_z become the fluxes at P in the directions of the x, y, z axes respectively. Equation (3) then reduces to

$$\lambda f_x + \mu f_y + \nu f_z = f \quad . \quad . \quad . \quad (4)$$

If the axes are chosen so that the flow of heat at P is parallel to the z-axis,

$$f_x = f_y = 0, \quad f_z = -\mathrm{K}\frac{\partial v}{\partial z},$$

and therefore $$f = \nu f_z = -\mathrm{K}\nu\frac{\partial v}{\partial z}.$$

But $$\frac{\partial v}{\partial n} = \lambda\frac{\partial v}{\partial x} + \mu\frac{\partial v}{\partial y} + \nu\frac{\partial v}{\partial z},$$

and, since the isothermal surface is parallel to the (x, y) plane $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$; hence

$$\frac{\partial v}{\partial n} = \nu\frac{\partial v}{\partial z},$$

so that

$$f = -\mathrm{K}\frac{\partial v}{\partial n}. \quad . \quad . \quad . \quad (5)$$

In particular

$$f_x = -K\frac{\partial v}{\partial x}, \quad f_y = -K\frac{\partial v}{\partial y}, \quad f_z = -K\frac{\partial v}{\partial z}.$$

§ 3. **The Equation of Conduction.** In proving the differential equation of conduction of heat use will be made of the following well-known theorem, by means of which a surface integral can be expressed in terms of volume integrals.

Green's Theorem. Let U and V be two functions, which, with their first and second derivatives, are uniform and continuous at all points of a space Σ bounded by a surface S: then, if n is the normal to the surface, measured outwards,

$$\iint U\frac{\partial V}{\partial n}dS = \iiint\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)d\Sigma$$
$$+ \iiint U\nabla^2 V d\Sigma, \qquad (6)$$

where $\nabla^2 V$ denotes the expression

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2},$$

and the surface and volume integrals are taken over S and throughout Σ respectively.

For, since a line drawn parallel to the x-axis meets the surface in an even number of points, on integrating by parts with regard to x we obtain

$$\iiint\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}dxdydz$$
$$= \iint\left[-\left(U\frac{\partial V}{\partial x}\right)_1 + \left(U\frac{\partial V}{\partial x}\right)_2 - \left(U\frac{\partial V}{\partial x}\right)_3 + \ldots\right]dydz$$
$$- \iiint U\frac{\partial^2 V}{\partial x^2}dxdydz,$$

where $\left(U\frac{\partial V}{\partial x}\right)_r$ is the value of $\left(U\frac{\partial V}{\partial x}\right)$ at the rth intersection of the line with the surface Hence, if (λ, μ, ν) are the direction-cosines of n, since $dydz$ is $-\lambda dS$ if r is odd, but λdS if r is even,

$$\iiint\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}d\Sigma = \iint U\frac{\partial V}{\partial x}\lambda dS - \iiint U\frac{\partial^2 V}{\partial x^2}d\Sigma.$$

Similarly

$$\iiint \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} d\Sigma = \iint U\frac{\partial V}{\partial y}\mu dS - \iiint U\frac{\partial^2 V}{\partial y^2} d\Sigma,$$

and

$$\iiint \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} d\Sigma = \iint U\frac{\partial V}{\partial z}\nu dS - \iiint U\frac{\partial^2 V}{\partial z^2} d\Sigma.$$

On adding these three equations, equation (6) is obtained, since

$$\lambda\frac{\partial V}{\partial x} + \mu\frac{\partial V}{\partial y} + \nu\frac{\partial V}{\partial z} = \frac{\partial V}{\partial n}.$$

From this result the equation of conduction can be deduced as follows. Let Σ be any volume of the solid, bounded by the surface S; then the amount of heat flowing into the solid per unit of time is

$$-\iint f dS = \iint K\frac{\partial v}{\partial n} dS,$$

and this is equal to the gain of heat

$$\iiint cp\frac{\partial v}{\partial t} d\Sigma$$

by the solid per unit of time; hence

$$\iiint \frac{\partial v}{\partial t} d\Sigma = \frac{K}{cp}\iint \frac{\partial v}{\partial n} dS.$$

Now, if U = 1 and V = v, equation (6) becomes

$$\iint \frac{\partial v}{\partial n} dS = \iiint \nabla^2 v d\Sigma,$$

so that

$$\iiint \frac{\partial v}{\partial t} d\Sigma = \frac{K}{cp}\iiint \nabla^2 v d\Sigma.$$

But this identity is true no matter how small the volume Σ may be: thus, at all points of the solid

$$\frac{\partial v}{\partial t} = \kappa\nabla^2 v = \kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad . \quad . \quad (7)$$

where $\kappa = \frac{K}{cp}$. Equation (7) is the *Equation of Conduction*, and the constant κ is called the *Diffusivity* or the *Thermometric Conductivity* of the substance.

§ 4. **Initial and Boundary Conditions.** The value of v obtained as a solution of (7) for a given solid must be a continuous function of x, y, z, and t, and this must also be true of its first derivative with regard to t and of its first and second derivatives with regard to x, y, and z.

In order that v may be completely determined, there must be given certain boundary or surface conditions, and also the initial conditions; that is, the values of v throughout the body when $t = 0$.

If the initial distribution of temperature is a continuous function, v must converge to this value when t tends to zero. If the initial distribution is discontinuous at a finite number of points or surfaces, these discontinuities will disappear very rapidly; when t tends to zero, v must converge to this initial value at all points where the initial distribution is continuous.

There are several possible types of Boundary or Surface Conditions.

(i) At the surface of a body placed in a medium heat is usually lost or gained by radiation and by convection. Let v_0 be the temperature of the medium, and v that of a small area δS of the surface: then it can be assumed, as a result of experiment, that the amount of heat passing off the area in time δt is

$$e(v - v_0)\delta S \delta t,$$

where e is a constant called the *Emissivity* of the surface. As a matter of fact, it is found that e does vary considerably with the temperature of the surface, so that, in experiments on conduction it is best to cut down the loss of heat by radiation as far as possible by treating the surface with a suitable material.

Now the amount of heat that passes through the area δS in time δt is

$$-K\frac{\partial v}{\partial n}\delta S \delta t,$$

and therefore

$$-K\frac{\partial v}{\partial n} = e(v - v_0).$$

Thus the surface condition is

$$\frac{\partial v}{\partial n} + h(v - v_0) = 0, \quad . \quad . \quad . \quad (8)$$

where $k = e/K$, and the differentiation is along the outward-drawn normal. If (λ, μ, ν) are the direction-cosines of this normal, (8) may be written

$$\lambda\frac{\partial v}{\partial x} + \mu\frac{\partial v}{\partial y} + \nu\frac{\partial v}{\partial z} + h(v - v_0) = 0 \quad . \quad . \quad (8')$$

(ii) If the surface be maintained at a constant temperature v_0, the surface condition is simply

$$v = v_0. \qquad (9)$$

(iii) If the surface be made impermeable to heat, so that no heat passes across it, the surface condition is

$$\frac{\partial v}{\partial n} = 0, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or

$$\lambda\frac{\partial v}{\partial x} + \mu\frac{\partial v}{\partial y} + \nu\frac{\partial v}{\partial z} = 0 \quad . \quad . \quad . \quad (10')$$

at the boundary.

(iv) If two solid bodies are in contact, then if v and v' are their temperatures at their common boundary,

$$K\frac{\partial v}{\partial n} = K'\frac{\partial v'}{\partial n}, \quad . \quad . \quad . \quad (11)$$

where K and K′ are the conductivities of the respective bodies, and the normal n is measured in the same direction for both. As a rule the additional equation

$$v = v' \quad . \quad . \quad . \quad . \quad . \quad (12)$$

will also hold at the common boundary.

It may be that one of the above boundary conditions holds over one part of the surface, and another over another part.

§ 5. **Uniqueness of the Solution.** If a solution of (7) can be found which satisfies the conditions stated in § 4, that solution will be unique. The following proof of this theorem holds when v or $\frac{\partial v}{\partial n}$ is given at all points of the boundary.

Suppose that there are two possible solutions V_1 and V_2, and let $v = V_1 - V_2$; then v satisfies (7), it has the value zero when $t = 0$, and either v or $\frac{\partial v}{\partial n}$ is zero at each point of the boundary.

Now let

$$J = \iiint \tfrac{1}{2} v^2 d\Sigma,$$

the integral being taken throughout the body; then

$$\frac{\partial J}{\partial t} = \iiint v \frac{\partial v}{\partial t} d\Sigma = \kappa \iiint v \nabla^2 v d\Sigma,$$

by (7). Now, in (6), let $U = v$, $V = v$, and it gives

$$\frac{\partial J}{\partial t} = \kappa \iint v \frac{\partial v}{\partial n} dS - \kappa \iiint \left\{ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} d\Sigma.$$

But, on the boundary, either $v = 0$ or $\frac{\partial v}{\partial n} = 0$, so that the first of these integrals is zero: hence

$$\frac{\partial J}{\partial t} = -\kappa \iiint \left\{ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} d\Sigma.$$

Thus, since the integrand cannot be negative, $\frac{\partial J}{\partial t}$ is either negative or zero: but, when $t = 0$, $v = 0$, so that $J = 0$; therefore, when $t \neq 0$,

$$J \leqq 0.$$

On the other hand, since

$$J = \iiint \tfrac{1}{2} v^2 d\Sigma,$$

J cannot be negative, and therefore J, and consequently v, must vanish for all values of t. Hence, finally, V_1 and V_2 are identically equal, which proves the theorem.

§ 6. **Infinite Slab with Parallel Faces.** Consider a slab of homogeneous material bounded by the planes $x = 0$, $x = l$; let the temperature be the same for all sections of the slab parallel to these two faces, so that it is a function of x alone: the problem may therefore be regarded as being one-dimensional. The faces are kept at zero temperature and the initial temperature is $v = f(x)$. Thus the problem is reduced to the solution of the equation

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l, \qquad . \quad . \quad (13)$$

with $\quad v = 0$, when $x = 0$ and when $x = l$,

and $\quad v = f(x)$, when $t = 0$.

In (13) put $v = \mathrm{TX}$, where T and X are functions of t and x alone, and divide by $\kappa\mathrm{TX}$; the equation then becomes

$$\frac{1}{\kappa}\,\frac{\dfrac{d\mathrm{T}}{dt}}{\mathrm{T}} = \frac{\dfrac{d^2\mathrm{X}}{dx^2}}{\mathrm{X}},$$

in which the left-hand side is independent of x, and the right-hand side of t; thus each side is independent of t and x, and has therefore a constant value, C say. Accordingly

$$\frac{d\mathrm{T}}{dt} = \kappa\mathrm{CT}, \quad \frac{d^2\mathrm{X}}{dx^2} = \mathrm{CX};$$

whence $\mathrm{T} = e^{\kappa\mathrm{C}t}, \quad \mathrm{X} = e^{\pm x\sqrt{(\mathrm{C})}},$

and $v = e^{\kappa\mathrm{C}t}\{\mathrm{A}e^{\sqrt{(\mathrm{C})}x} + \mathrm{B}e^{-\sqrt{(\mathrm{C})}x}\},$

A and B being arbitrary constants.

When $x = 0$, $v = 0$, and consequently

$$\mathrm{A} + \mathrm{B} = 0,$$

so that

$$v = \mathrm{A}e^{\kappa\mathrm{C}t}\{e^{\sqrt{(\mathrm{C})}x} - e^{-\sqrt{(\mathrm{C})}x}\}.$$

Now, in order that v may be zero when $x = l$ and may remain finite when t tends to infinity, $\sqrt{\mathrm{C}}$ must be imaginary, making C negative, and $\sqrt{(\mathrm{C})}\,l$ must be of the form $in\pi$, where n is an integer; hence

$$v = \mathrm{C}_n e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l},$$

where C_n is an arbitrary constant, and the most general solution obtained in this way is

$$v = \sum_{n=1}^{\infty}\mathrm{C}_n e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l},$$

provided that the constants are chosen so that the series is convergent.

Now, if $f(x)$ satisfies Dirichlet's Conditions (§ 2), the sine series for $f(x)$ for $0 < x < l$ is

$$f(x) = \sum_{n=1}^{\infty}\mathrm{C}_n\sin\frac{n\pi x}{l},$$

where $$\mathrm{C}_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy.$$

Hence

$$v = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy \quad . \quad (14)$$

satisfies the conditions of the problem, provided that the series is convergent, and that it can be differentiated with regard to t and x.

Now, if $t \geqq \tau$ where τ is positive, since (Ch. I., § 4, Note 2),

$$| C_n | < \frac{C}{n},$$

where C is a definite constant, the series (14) and all the series derived from it by differentiating with regard to x and t are uniformly convergent in x and t for $0 \leqq x \leqq l$, $\tau \leqq t$. The value of v given by (14) may therefore be differentiated with regard to t and x, and is consequently a solution of (13). Also, when t tends to zero v tends to $f(x)$ for all values of x for which $f(x)$ is continuous.

Finite Rod. This solution also holds for a finite rod of length l and of uniform cross-section, with $v = 0$ at $x = 0$ and at $x = l$, and with $v = f(x)$ when $t = 0$, provided that there is no radiation at the surface of the rod.

Ends at Fixed Temperatures. If in the slab or rod the ends, instead of being at temperature zero, have fixed temperatures v_1 and v_2, the equations to be satisfied are

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= \kappa\frac{\partial^2 v}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l, \\ v &= v_1, \text{ when } x = 0, \\ v &= v_2, \text{ when } x = l, \\ v &= f(x), \text{ when } t = 0. \end{aligned}\right\} \quad (15)$$

Here let $$v = v_1 + (v_2 - v_1)\frac{x}{l} + u;$$

then u satisfies the equation

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l,$$

with $$u = 0, \text{ when } x = 0 \text{ and when } x = l,$$

and $$u = f(x) - v_1 - (v_2 - v_1)\frac{x}{l}, \text{ when } t = 0,$$

and this is of the same type as (13). Thus, from (14),

$$u = \sum_{n=1}^{\infty} C_n e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l},$$

where
$$C_n = \frac{2}{l}\int_0^l \left\{f(y) - v_1 - (v_2 - v_1)\frac{y}{l}\right\} \sin\frac{n\pi y}{l}\, dy$$
$$= \frac{2}{l}\int_0^l f(y) \sin\frac{n\pi y}{l}\, dy + \frac{2}{n\pi}(v_2 \cos n\pi - v_1).$$

Hence the solution of (15) is

$$v = v_1 + (v_2 - v_1)\frac{x}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{v_2\cos n\pi - v_1}{n}\sin\frac{n\pi x}{l}\, e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$
$$+ \frac{2}{l}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\, e^{-\kappa\frac{n^2\pi^2}{l^2}t}\int_0^l f(y)\sin\frac{n\pi y}{l}\, dy. \quad (16)$$

§ 7. **Radiation at the Surface of a Rod of Small Cross-Section.** It is assumed that the cross-section of the rod is uniform throughout, and that the temperature is the same at each point of the same cross-section : σ is the area and p the perimeter of a cross-section, and x is the distance of a point P on the rod from one end. At time t the amount of heat per sec. passing through the cross-section at P is $-\text{K}\frac{\partial v}{\partial x}\sigma$, and the amount through the cross-section with abscissa $x + \delta x$, where δx is small, is $-\text{K}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\delta x\right)\sigma$; hence the total amount entering the small volume between these sections is

$$\text{K}\frac{\partial^2 v}{\partial x^2}\delta x\sigma.$$

If the temperature of the surrounding medium is zero, the rate at which the surface of the volume is losing heat by radiation is

$$evp\delta x$$

per second, so that the total heat gained by the volume per second is

$$\text{K}\frac{\partial^2 v}{\partial x^2}\delta x\sigma - evp\delta x.$$

Now the rate at which the volume is gaining heat is

$$c\rho\frac{\partial v}{\partial t}\sigma\delta x;$$

therefore
$$\frac{\partial v}{\partial t} = \frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2} - \frac{ep}{c\rho\sigma}v,$$

or
$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - hv, \quad . \quad . \quad . \quad (17)$$

where $\kappa = K/c\rho$ and $h = ep/(c\rho\sigma)$; κ, of course, as in § 3, is the Diffusivity.

If the temperature of the surrounding medium is v_0, equation (17) becomes

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - h(v - v_0). \quad . \quad . \quad . \quad (18)$$

When the surface is impervious to heat, e and h are zero, and equation (17) is then the same as (13). Even when there is radiation, equation (17) can be reduced to the same form as (13) by the transformation

$$v = ue^{-ht},$$

which gives
$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The corresponding transformation for (18) is

$$v = v_0 + ue^{-ht}.$$

If the cross-section of the rod is very small, these results, and also those of § 6, hold even if the rod is curved, x being then measured along the rod.

Rod with Ends at Temperature Zero. Let the conditions to be satisfied be

$$\begin{aligned} &\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - hv, \text{ if } t > 0,\ 0 < x < l, \\ &v = 0, \text{ when } x = 0 \text{ and when } x = l, \qquad (20) \\ &v = f(x), \text{ when } t = 0. \end{aligned}$$

Then, if $v = ue^{-ht}$, u satisfies the equation and conditions (13), and therefore, from (14) the solution is

$$v = \frac{2}{l}e^{-ht}\sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy. \quad (21)$$

§ 8. Fourier's Ring. The first problem to which Fourier applied his theory was that of the conduction of heat in a circular ring of small uniform cross-section. The appropriate differential equations are then the same as those of sections 6 and 7, but the solution must be such that the value of v for any value of x will be equal to its value for $x + nl$, where n is any integer and l is the length of the ring.

Case I. No Radiation. Suppose that the radius of the ring is unity, and let there be no radiation at the surface. Then, if the initial temperature is $f(x)$, the conditions to be satisfied are

$$\begin{aligned} \frac{\partial v}{\partial t} &= \kappa\frac{\partial^2 v}{\partial x^2}, \text{ if } t > 0, \ -\pi \leqq x \leqq \pi, \\ v &= f(x), \text{ if } t = 0, \ -\pi \leqq x \leqq \pi, \\ v_x &= v_{x+2n\pi}, \text{ if } t > 0. \end{aligned} \tag{22}$$

As in § 6 it can be shown that the differential equation has a solution of the form

$$v = e^{-\kappa C^2 t}(A \cos Cx + B \sin Cx),$$

and, since v has a period 2π, C must be an integer. The most general solution obtained in this way is

$$v = A_0 + \sum_{n=1}^{\infty} e^{-\kappa n^2 t}(A_n \cos nx + B_n \sin nx).$$

When t is zero, this becomes

$$f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

and consequently

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy,$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \cos ny\, dy, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \sin ny\, dy.$$

Hence, finally,

$$v = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-\kappa n^2 t}\int_{-\pi}^{\pi} f(y) \cos n(y - x)\, dy. \tag{23}$$

The convergence of the series can be discussed in the same manner as that of the series in (14).

Case II. Steady Temperature. Suppose that the section of the ring at $x = \pm \pi$ is maintained at a temperature V until the flow of heat has become constant, and that radiation is taking place into a medium at temperature zero. The equations to be satisfied are, from (17),

$$\frac{d^2v}{dx^2} = \mu^2 v, \text{ where } \mu^2 = \frac{h}{\kappa}, \quad -\pi < x < \pi, \qquad . \quad (24)$$

$$v = V, \text{ at } x = \pm \pi,$$

and $$\frac{dv}{dx} = 0, \text{ at } x = 0,$$

the last equation being required owing to the symmetry of the distribution of temperature about the diameter through the points $x = 0$ and $x = \pi$.

The general solution of (24) is

$$v = Ae^{\mu x} + Be^{-\mu x},$$

and, since $\frac{dv}{dx} = 0$ when $x = 0$, A and B must be equal; hence

$$v = 2A \cosh \mu x.$$

Also, since $v = V$ when $x = \pm \pi$,

$$V = 2A \cosh \mu\pi.$$

Therefore the solution of the problem is

$$v = V \frac{\cosh \mu x}{\cosh \mu\pi}. \qquad . \quad . \quad . \quad (25)$$

Case III. Cooling due to Radiation. Suppose the ring to have been heated to the condition discussed in *Case II.*, and that the source of heat is then removed, and the ring allowed to cool. The temperature of the surrounding medium is assumed to be zero, and the moment at which the source of heat is removed is taken to be $t = 0$. The equations to be satisfied are

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - hv, \text{ if } t > 0, \quad -\pi \leqq x \leqq \pi, \qquad . \quad (26)$$

$$v = V \frac{\cosh \mu x}{\cosh \mu\pi}, \text{ if } t = 0, \quad -\pi \leqq x \leqq \pi,$$

$$v_x = v_{x+2n\pi}, \text{ if } t > 0,$$

where $$\mu = \sqrt{\left(\frac{h}{\kappa}\right)}.$$

In (26) put $v = ue^{-ht}$; then the equations to be satisfied by u are

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \text{ if } t > 0, \ -\pi \leqq x \leqq \pi,$$

$$u = \mathrm{V}\frac{\cosh \mu x}{\cosh \mu\pi}, \text{ if } t = 0, \ -\pi \leqq x \leqq \pi,$$

$$u_x = u_{x+2n\pi}, \text{ if } t > 0.$$

Now (Ch. I., § 6, *ex.* 3),

$$\frac{\cosh \mu x}{\cosh \mu\pi} = \frac{2\mu \tanh \mu\pi}{\pi}\left\{\frac{1}{2\mu^2} + \sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{\mu^2+n^2}\right\},$$

and the value of u can be deduced from the solution in *Case I.*; hence

$$v = \frac{2\mathrm{V}\mu}{\pi}\tanh \mu\pi\, e^{-ht}\left\{\frac{1}{2\mu^2} + \sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{\mu^2+n^2}e^{-\kappa n^2 t}\right\}. \qquad (27)$$

§ 9. **Duhamel's Theorem.** If the surface temperature varies with the time, the solution of the equation of conduction can be obtained by means of the following theorems:—

Theorem I. Let the initial temperature of a body be zero, and the surface temperature $\phi(x, y, z, t)$; then, if the solution for the case in which the initial temperature is zero and the surface temperature is $\phi(x, y, z, \lambda)$, where λ is a constant, be

$$v = \mathrm{F}(x, y, z, \lambda, t),$$

the solution for the case in which the surface temperature is $\phi(x, y, z, t)$ is

$$v = \int_0^t \frac{\partial}{\partial t}\mathrm{F}(x, y, z, \lambda, t - \lambda)d\lambda.$$

For, suppose that the surface temperature has been zero from $t = 0$ to $t = \lambda$, where $\lambda < t$, and $\phi(x, y, z, \lambda)$ from $t = \lambda$ to $t = t$; then, since the initial temperature is zero, the temperature of the body at time λ will be zero, and this can be taken as the initial time. The solution in this case is therefore

$$v = \mathrm{F}(x, y, z, \lambda, t - \lambda), \ t > \lambda.$$

Again, if the surface temperature has been zero from $t = 0$ to $t = \lambda + \delta\lambda$, where $\lambda + \delta\lambda < t$, and $\phi(x, y, z, \lambda)$ from $t = \lambda + \delta\lambda$ to $t = t$, the temperature at time t will be

$$v - \delta v = \mathrm{F}(x, y, z, \lambda, t - \lambda - \delta\lambda), \ t > \lambda + \delta\lambda.$$

Now, if the second of these solutions be subtracted from the first, we see that, if the surface temperature has been zero from $t = 0$ to $t = \lambda$, $\phi(x, y, z, \lambda)$ from $t = \lambda$ to $t = \lambda + \delta\lambda$, and zero from $t = \lambda + \delta\lambda$ to $t = t$, and the initial temperature is zero, the temperature at time t will be

$$\delta v = \mathrm{F}(x, y, z, \lambda, t - \lambda) - \mathrm{F}(x, y, z, \lambda, t - \lambda - \delta\lambda),$$

or, if $\delta\lambda$ tends to zero,

$$\frac{\partial v}{\partial \lambda} = \frac{\partial}{\partial t}\mathrm{F}(x, y, z, \lambda, t - \lambda), \quad t > \lambda.$$

Hence the solution at time t due to initial temperature zero and surface temperature $\phi(x, y, z, t)$ is

$$v = \int_0^t \frac{\partial}{\partial t}\mathrm{F}(x, y, z, \lambda, t - \lambda)d\lambda.$$

When the surface temperature $\phi(t)$ is the same at all points of the boundary, this theorem can be stated in the following somewhat simpler form :—

Theorem II. If the solution when the initial temperature is zero and the surface temperature is unity is

$$v = \mathrm{F}(x, y, z, t),$$

the solution when the initial temperature is zero and the surface temperature is $\phi(t)$ is

$$v = \int_0^t \phi(\lambda)\frac{\partial}{\partial t}\mathrm{F}(x, y, z, t - \lambda)d\lambda.$$

The proof, which is on the same lines as that of *Theorem I.*, is left as an exercise to the reader.

Similar theorems hold for radiation into a medium whose temperature is $\phi(x, y, z, t)$ or $\phi(t)$.

For the more general problem in which $v = f(x, y, z)$ initially and $v = \phi(x, y, z, t)$ on the boundary, put

$$v = u + w,$$

where $u = f(x, y, z)$ initially and $u = 0$ on the boundary, while $w = 0$ initially and $w = \phi(x, y, z, t)$ on the boundary; Duhamel's Theorem is then employed to obtain w.

§ 10. **Finite Rod with Variable End Temperatures.**

First of all, suppose that there is no radiation at the surface. Then the equations to be satisfied are

$$\begin{aligned} \frac{\partial v}{\partial t} &= \kappa\frac{\partial^2 v}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l, \\ v &= \phi_1(t), \text{ when } x = 0, \\ v &= \phi_2(t), \text{ when } x = l, \\ \text{and} \qquad v &= f(x), \text{ when } t = 0, \end{aligned} \qquad (28)$$

Now, let $v = u + w$, where

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa\frac{\partial^2 u}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l, \\ u &= 0, \text{ when } x = 0 \text{ and when } x = l, \\ u &= f(x) \text{ when } t = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial w}{\partial t} &= \kappa\frac{\partial^2 w}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l, \\ w &= \phi_1(t), \text{ when } x = 0, \\ w &= \phi_2(t), \text{ when } x = l, \\ w &= 0, \quad \text{ when } t = 0. \end{aligned}$$

From (13) and (14) it follows that

$$u = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy,$$

and from (15) and (16) that

$$\begin{aligned} F(x, \lambda, t) = {} & \phi_1(\lambda)\left\{1 - \frac{x}{l} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}\right\} \\ & + \phi_2(\lambda)\left\{\frac{x}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}\right\}. \end{aligned}$$

Thus

$$\begin{aligned} w &= \int_0^t \frac{\partial}{\partial t}F(x, \lambda, t - \lambda)d\lambda \\ &= \frac{2\kappa\pi}{l^2}\sum_{n=1}^{\infty} n e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}\int_0^t \phi_1(\lambda)e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}d\lambda \\ &\quad - \frac{2\kappa\pi}{l^2}\sum_{n=1}^{\infty}(-1)^n n e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}\int_0^t \phi_2(\lambda)e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}d\lambda. \end{aligned}$$

Hence, finally, if $0 < x < l$,

$$v = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}\left[\int_0^l f(y)\sin\frac{n\pi y}{l}dy\right.$$

$$\left.+\frac{n\pi\kappa}{l}\int_0^t e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}\{\phi_1(\lambda)-(-1)^n\phi_2(\lambda)\}d\lambda\right] \quad (29)$$

Finite Rod with Ends at Fixed Temperatures and Radiation at the Surface. The equations satisfied by the temperature v are

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - hv, \text{ if } t > 0,\ 0 < x < l,$$

$$v = v_1, \quad \text{when } x = 0,$$

$$v = v_2, \quad \text{when } x = l,$$

$$v = f(x), \text{ when } t = 0,\ 0 < x < l.$$

Here let $v = ue^{-ht}$, and u will satisfy

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \text{ if } t > 0,\ 0 < x < l,$$

$$u = v_1 e^{ht}, \text{ when } x = 0,$$

$$u = v_2 e^{ht}, \text{ when } x = l,$$

$$u = f(x), \text{ when } t = 0,\ 0 < x < l.$$

These equations are of the same form as (28), and therefore, employing the formula (29), we find that

$$v = 2e^{-ht}\sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}\left[\frac{1}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy\right.$$

$$\left.+\,n\kappa\pi\frac{v_1-(-1)^n v_2}{\kappa n^2\pi^2+l^2h}\left\{e^{\left(\kappa\frac{n^2\pi^2}{l^2}+h\right)t}-1\right\}\right], \quad (30)$$

where $0 < x < l$.

§ 11. **Steady Temperature in an Infinite Rectangular Solid.** Consider the solid bounded by the planes $x = 0, x = \pi$, which are kept at zero temperature, and the plane $y = 0$, which is kept at steady temperature $v = f(x)$. The equations to be satisfied are

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \text{ if } 0 < x < \pi,\ 0 < y, \quad . \quad (31)$$

$$v = 0, \text{ when } x = 0 \text{ and when } x = \pi,$$

$$v = f(x), \text{ when } y = 0,\ 0 < x < \pi,$$

$$\lim_{y\to\infty} v = 0.$$

In (31) put $v = XY$, where X and Y are functions of x and y alone; then, dividing by XY, we get

$$-\frac{\dfrac{d^2X}{dx^2}}{X} = \frac{\dfrac{d^2Y}{dy^2}}{Y},$$

and, as in § 6, we can show that each of these fractions is equal to a constant C: thus

$$\frac{d^2X}{dx^2} = -CX, \quad \frac{d^2Y}{dy^2} = CY,$$

so that $$X = Ae^{\sqrt{(-C)}x} + Be^{-\sqrt{(-C)}x},$$

and $$Y = e^{\pm\sqrt{(C)}y}.$$

Since v and therefore X vanish when $x = 0$, $B = -A$:

thus $$v = e^{\pm\sqrt{(C)}y} \sin\{\sqrt{(C)}x\}$$

is a solution, and, since it must vanish when $x = \pi$, $\sqrt{(C)}$ must be an integer, n say. Also, since v tends to zero when y tends to infinity, we must, when n is positive, take the upper sign in the expression for Y. Thus the solution is

$$v = e^{-ny} \sin nx,$$

and the most general solution obtained in this way is

$$v = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx,$$

where A_n is arbitrary, subject to the condition that the series must be convergent.

Now, when $y = 0$,

$$v = f(x) = \sum_{n=1}^{\infty} A_n \sin nx.$$

Hence $$A_n = \frac{2}{\pi}\int_0^{\pi} f(y) \sin ny \, dy,$$

and therefore the solution is

$$v = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-ny} \sin nx \int_0^{\pi} f(y) \sin ny \, dy, \quad 0 \leqq x \leqq \pi, \ 0 \leqq y. \quad (32)$$

The convergence of the series can be discussed in the same way as that of the series in (14).

§ 12. **Rectangular Parallelepiped: Steady Temperature.** Consider the solid bounded by the planes $x = 0, x = a, y = 0, y = b, z = 0, z = c$; the boundary conditions are

$$v = v_1, \text{ when } x = 0,$$
$$v = v_2, \text{ when } x = a,$$

with the other faces at zero temperature. Also, since the temperature is steady, the equation of conduction is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \quad . \quad . \quad . \quad (33)$$

Here let $v = XYZ$, where X, Y, Z are functions of x, y, z alone; then, substituting in (33), and dividing by XYZ, we find that

$$\frac{\frac{d^2X}{dx^2}}{X} + \frac{\frac{d^2Y}{dy^2}}{Y} + \frac{\frac{d^2Z}{dz^2}}{Z} = 0 \quad . \quad . \quad . \quad (34)$$

Now, since the second and third of these terms are independent of x, so also is the first. It is therefore constant in value, and we may write

$$\frac{d^2X}{dx^2} = C_1X.$$

Similarly

$$\frac{d^2Y}{dy^2} = C_2Y, \quad \frac{d^2Z}{dz^2} = C_3Z,$$

where, from (34),

$$C_1 + C_2 + C_3 = 0. \qquad (35)$$

Solving the equation for Y, we find that

$$Y = Ae^{\sqrt{(C_2)}} + Be^{-\sqrt{(C_2)}};$$

but, since $Y = 0$ when $y = 0$, $B = -A$; hence

$$Y = A\{e^{\sqrt{(C_2)}y} - e^{-\sqrt{(C_2)}y}\}.$$

Again, $Y = 0$ when $y = b$, and therefore $\sqrt{C_2}$ must be of the form $im\pi/b$, where m is an integer: hence

$$Y = C \sin\frac{m\pi y}{b}.$$

Similarly it can be shown that $\sqrt{C_3} = in\pi/c$, where n is an integer, and that

$$Z = D\sin\frac{n\pi z}{c}.$$

Equation (35) then becomes

$$C_1 = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{c^2}\right),$$

so that C_1 must be real and positive. Let $C_1 = l^2$, and the equation for X becomes

$$\frac{d^2X}{dx^2} = l^2X,$$

and has the solution

$$X = Ee^{lx} + Fe^{-lx}.$$

The most general solution obtained in this way is

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{E_{m,n}e^{lx} + F_{m,n}e^{-lx}\}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}, \quad (36)$$

where

$$l^2 = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{c^2}\right).$$

Now, when $x = 0$, $v = v_1$, and when $x = a$, $v = v_2$; hence, from (36),

$$v_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{E_{m,n} + F_{m,n}\}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}, \quad (37)$$

$$v_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{E_{m,n}e^{la} + F_{m,n}e^{-la}\}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}. \quad (38)$$

But, if $0 < y < b$, $0 < z < c$,

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi y}{b} + \tfrac{1}{3}\sin\frac{3\pi y}{b} + \tfrac{1}{5}\sin\frac{5\pi y}{b} + \ldots\right),$$

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi z}{c} + \tfrac{1}{3}\sin\frac{3\pi z}{c} + \tfrac{1}{5}\sin\frac{5\pi z}{c} + \ldots\right),$$

and therefore, on multiplying, we find that

$$1 = \frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2p+1}\,\frac{1}{2q+1}\sin\frac{(2p+1)\pi y}{b}\sin\frac{(2q+1)\pi z}{c}.$$

On comparing (37) and (38) with this equation, we see that $E_{m,n}$ and $F_{m,n}$ both vanish unless m and n are both odd, and that, when $m = 2p + 1$, $n = 2q + 1$

$$\frac{E_{m,n} + F_{m,n}}{v_1} = \frac{16}{\pi^2}\,\frac{1}{2p+1}\,\frac{1}{2q+1},$$

$$\frac{E_{m,n}e^{la} + F_{m,n}e^{-la}}{v_2} = \frac{16}{\pi^2}\,\frac{1}{2p+1}\,\frac{1}{2q+1}.$$

On solving these equations, it is found that

$$E_{m,n} = \frac{16}{\pi^2}\,\frac{1}{2p+1}\,\frac{1}{2q+1}\,\frac{v_2 - v_1e^{-la}}{2\sinh(la)},$$

$$F_{m,n} = \frac{16}{\pi^2}\,\frac{1}{2p+1}\,\frac{1}{2q+1}\,\frac{v_1e^{la} - v_2}{2\sinh(la)}.$$

Thus (36) becomes

$$v = \frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{v_1\sinh l(a-x) + v_2\sinh(lx)}{\sinh(la)} \times \frac{\sin\dfrac{(2p+1)\pi y}{b}}{2p+1}\,\frac{\sin\dfrac{(2q+1)\pi z}{c}}{2q+1}, \qquad (39)$$

where
$$\frac{l^2}{\pi^2} = \frac{(2p+1)^2}{b^2} + \frac{(2q+1)^2}{c^2}.$$

§ 13. **Rectangular Parallelepiped: Variable Temperature.** Suppose that all the faces of the solid considered in the last section are at temperature zero, and let there be no radiation at the surfaces, then

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \qquad (40)$$

$$v = 0, \text{ when } x = 0,\ y = 0,\ z = 0,$$
$$x = a,\ y = b,\ z = c, \qquad (41)$$

and
$$v = f(x, y, z) \text{ when } t = 0 \qquad (42)$$

By putting $v = TXYZ$, where T, X, Y, Z are functions of t, x, y, z alone, it can be shown, in the same manner as in the previous section, that

$$e^{-\kappa\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c},$$

where l, m, n are integers, satisfies (40) and (41). The most general solution obtained in this way is

$$v = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{l,m,n} e^{-\kappa\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c},$$

and, when $t = 0$, this gives

$$f(x, y, z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{l,m,n} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}.$$

If we assume that an expansion of the Fourier type is valid in this case, it follows that

$$A_{l,m,n} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(\xi, \eta, \zeta) \sin\frac{l\pi\xi}{a} \sin\frac{m\pi\eta}{b} \sin\frac{n\pi\zeta}{c} d\zeta d\eta d\xi.$$

In particular, if $f(x, y, z) = 1$,

$$A_{l,m,n} = \frac{8}{lmn\pi^3}(1 - \cos l\pi)(1 - \cos m\pi)(1 - \cos n\pi),$$

and

$$v = \left(\frac{4}{\pi}\right)^3 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} e^{-\kappa\pi^2\left\{\frac{(2p+1)^2}{a^2}+\frac{(2q+1)^2}{b^2}+\frac{(2r+1)^2}{c^2}\right\}t}$$
$$\times \frac{\sin\frac{(2p+1)\pi x}{a} \sin\frac{(2q+1)\pi y}{b} \sin\frac{(2r+1)\pi z}{c}}{(2p+1)(2q+1)(2r+1)}, \quad (43)$$

where $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, $0 \leqq z \leqq c$.

§ 14. **Infinite Rod: No Radiation.** In the case of an infinite rod, with no radiation at the surface, the equations to be satisfied are

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \text{ if } t > 0, -\infty < x < \infty \quad . \quad . \quad (44)$$

$$v = f(x), \text{ when } t = 0, -\infty < x < \infty \quad . \quad . \quad (45)$$

In the same manner as in § 6, it can be shown that a solution of (44) is

$$v = e^{\kappa Ct \pm \sqrt{(C)}x},$$

where C is a constant. As the solution must be finite for all values of x, $\sqrt{C}$ must be imaginary; thus C is negative, so that, when t tends to infinity, v tends to zero. Two solutions then are

$$e^{-\kappa\beta^2 t}\cos\beta x, \quad e^{-\kappa\beta^2 t}\sin\beta x,$$

where β is an arbitrary constant. It follows that

$$\int_{-\infty}^{\infty} e^{-\kappa\beta^2 t}\cos\beta(x-\alpha)d\beta$$

or*

$$\sqrt{\left(\frac{\pi}{\kappa t}\right)}e^{-\frac{(x-\alpha)^2}{4\kappa t}}$$

is also a solution, where α is arbitrary; hence, if $\psi(\alpha)$ is any function, another solution is

$$v = \frac{1}{\sqrt{(t)}}\int_{-\infty}^{\infty} e^{-\frac{(x-\alpha)^2}{4\kappa t}}\psi(\alpha)d\alpha.$$

Here put $\alpha - x = 2\sqrt{(\kappa t)}\,\xi$, and the solution becomes

$$v = 2\sqrt{\kappa}\int_{-\infty}^{\infty} e^{-\xi^2}\psi\{x + 2\sqrt{(\kappa t)}\,\xi\}d\xi.$$

Now, when t tends to zero, this integral tends to

$$2\sqrt{(\kappa)}\psi(x)\int_{-\infty}^{\infty} e^{-\xi^2}d\xi = 2\sqrt{(\kappa\pi)}\,\psi(x).$$

Hence, in order that v may have the value $f(x)$ when $t = 0$, we must take $\frac{1}{2\sqrt{(\kappa\pi)}}f(x)$ as the value of $\psi(x)$: thus the solution of (44) and (45) is

$$\begin{aligned} v &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2}f\{x + 2\sqrt{(\kappa t)}\xi\}d\xi \\ &= \frac{1}{2\sqrt{(\kappa\pi t)}}\int_{-\infty}^{\infty} e^{-\frac{(x-\alpha)^2}{4\kappa t}}f(\alpha)d\alpha. \quad . \quad . \quad (46)\end{aligned}$$

Since*

$$\int_0^{\infty} e^{-\kappa y^2 t}\cos y(\alpha - x)\,dy = \tfrac{1}{2}\sqrt{\left(\frac{\pi}{\kappa t}\right)}\,e^{-\frac{(x-\alpha)^2}{4\kappa t}}, \quad (47)$$

(46) can be written

$$v = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\alpha)d\alpha\int_0^{\infty}\cos y(\alpha - x)e^{-\kappa y^2 t}\,dy, \quad . \quad (48)$$

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* Gibson's Calculus, p. 469.

a form which is suggested by Fourier's Integral

$$f(x) = \frac{1}{\pi}\int_0^\infty dy \int_{-\infty}^\infty f(\alpha)\cos y(\alpha - x)\, d\alpha.$$

§ 15. **Infinite Solid.** For an infinite solid with variable temperature, the equations to be satisfied are

$$\frac{\partial v}{\partial t} = \kappa\nabla^2 v, \text{ if } t > 0, \quad . \quad . \quad . \quad (49)$$

and

$$v = f(x, y, z) \text{ for } t = 0. \quad . \quad . \quad (50)$$

As in the previous section, it can be shown that

$$v = \frac{1}{t^{\frac{3}{2}}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4\kappa t}}\psi(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma$$

satisfies (49); transforming this as before, we find that

$$v = 8\kappa^{\frac{3}{2}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(\xi^2+\eta^2+\zeta^2)}$$
$$\times\ \psi\{x + 2\xi\sqrt{(\kappa t)},\ y + 2\eta\sqrt{(\kappa t)},\ z + 2\zeta\sqrt{(\kappa t)}\}\, d\xi\, d\eta\, d\zeta,$$

and, in order that (50) may be satisfied, $\psi(x, y, z)$ must be replaced by

$$\frac{1}{(4\kappa\pi)^{\frac{3}{2}}}f(x, y, z).$$

Hence the required solution is

$$v = \frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(\xi^2+\eta^2+\zeta^2)}$$
$$\times\ f\{x + 2\xi\sqrt{(\kappa t)},\ y + 2\eta\sqrt{(\kappa t)},\ z + 2\zeta\sqrt{(\kappa t)}\}\, d\xi\, d\eta\, d\zeta$$
$$= \frac{1}{8(\kappa\pi t)^{\frac{3}{2}}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4\kappa t}} f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma. \quad (51)$$

Semi-infinite Solid. Let the solid be bounded by the plane $x = 0$, and lie on the side of this plane for which x is positive. It will be assumed that $v = 0$ when $x = 0$, and that $v = f(x)$ when $t = 0$. The solution can be derived from that of the infinite solid by assuming that, for negative values of x, $v = -f(-x)$, when $t = 0$: from these initial conditions it will necessarily follow that $v = 0$ when $x = 0$. Then, from (46),

$$v = \frac{1}{2\sqrt{(\kappa\pi t)}}\left[\int_0^\infty e^{-\frac{(x-\alpha)^2}{4\kappa t}} f(\alpha)d\alpha + \int_{-\infty}^0 e^{-\frac{(x-\alpha)^2}{4\kappa t}}\{-f(-\alpha)\}d\alpha\right]$$

$$= \frac{1}{2\sqrt{(\kappa\pi t)}}\int_0^\infty f(\alpha)\left\{e^{-\frac{(x-\alpha)^2}{4\kappa t}} - e^{-\frac{(x+\alpha)^2}{4\kappa t}}\right\}d\alpha. \quad . \quad . \quad (52)$$

By means of (47) this can be transformed into

$$v = \frac{1}{\pi}\int_0^\infty f(\alpha)d\alpha\int_0^\infty \{\cos y(\alpha - x) - \cos y(\alpha + x)\}e^{-\kappa y^2 t}\,dy$$

$$= \frac{2}{\pi}\int_0^\infty f(\alpha)d\alpha\int_0^\infty \sin \alpha y \sin xy\, e^{-\kappa y^2 t}\,dy; \quad . \quad . \quad (53)$$

a solution suggested by Fourier's Integral—

$$f(x) = \frac{2}{\pi}\int_0^\infty dy\int_0^\infty f(\alpha)\sin \alpha y \sin xy\, d\alpha.$$

Example. If a quantity of heat Q is instantaneously generated at time $t = 0$ at a point in a homogeneous infinite solid whose temperature is zero, and if the point is taken as the origin, show that, at time t,

$$v = \frac{Q}{8\rho c(\pi\kappa t)^{\frac{3}{2}}}e^{-\frac{r^2}{4\kappa t}},$$

where $r^2 = x^2 + y^2 + z^2$.

As shown above, v is a solution of the equation (49), and, when $t = 0$, v is infinite at the origin and zero elsewhere. Also, if we integrate throughout the solid, we find that the total quantity of heat at time t is

$$\int_0^\infty v\rho c 4\pi r^2 dr = \frac{\pi Q}{2(\pi\kappa t)^{\frac{3}{2}}}\int_0^\infty e^{-\frac{r^2}{4\kappa t}}r^2 dr;$$

and, on integration by parts, this becomes

$$\left[-\frac{Q}{\sqrt{(\pi\kappa t)}}e^{-\frac{r^2}{4\kappa t}}r\right]_0^\infty + Q\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\frac{r^2}{4\kappa t}}\frac{dr}{\sqrt{(4\kappa t)}} = Q.$$

Examples.

1. A bar of length l is heated so that both ends remain at zero temperature. If one end is taken as the origin, and the initial temperature is $v = cx(l - x)/l^2$, show that the temperature at x at time t is

$$v = \frac{4ce^{-ht}}{\pi^3}\sum_{n=1}^{\infty}\frac{1 - \cos n\pi}{n^3}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}.$$

2. In the case of an infinite rod with no radiation at the surface, the initial temperature is given by $v = (-1)^n V$ between $x = nc$ and $x = (n+1)c$, where n is zero or any positive or negative integer. Show that, if $t > 0$,

$$v = \frac{4V}{\pi} \sum_{p=0}^{\infty} \frac{\sin (2p+1)\frac{\pi x}{c}}{2p+1} e^{-\kappa(2p+1)^2 \frac{\pi^2}{c^2} t}.$$

3. A uniform bar of length a is heated so that three successive portions of length $\frac{1}{3}a$ are respectively at temperatures V_1, V_2, V_3; if there be no radiation, either at the side or at the ends, show that, when the origin is taken at one end,

$$v = \frac{V_1 + V_2 + V_3}{3} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{V_1 - V_2}{n} \sin \frac{n\pi}{3} + \frac{V_2 - V_3}{n} \sin \frac{2n\pi}{3} \right\} e^{-\kappa \frac{n^2\pi^2}{a^2} t} \cos \frac{n\pi x}{a},$$

where $0 \leqq x \leqq a$.

4. One end of a finite rod is kept for a long time at temperature v_0, there being surface radiation into a medium at zero temperature. A part whose extremities are distant b and $b + l$ from this end is then cut off from the rod and kept from loss or gain of heat. Show that the temperature at time t at a point distant x from the end of the part is

$$v_0 e^{-\mu b} \frac{1 - e^{-\mu l}}{\mu l} + \frac{2 v_0 \mu}{l} e^{-\mu b} \sum_{n=1}^{\infty} \frac{1 - e^{-\mu l} \cos n\pi}{\mu^2 + \frac{n^2\pi^2}{l^2}} e^{-\kappa \frac{n^2\pi^2}{l^2} t} \cos \frac{n\pi x}{l},$$

where $\mu^2 = h/\kappa$.

5. An infinite plate is bounded by the planes $x = 0$, $x = l$, which are kept at temperature zero, and by the plane $y = 0$, which is kept at steady temperature $v = c$. Show that, at any point of the plate

$$v = \frac{2c}{\pi} \tan^{-1} \left\{ \frac{\sin \frac{\pi x}{l}}{\sinh \frac{\pi y}{l}} \right\}.$$

6. A solid is bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = b$. If the planes $x = 0$, $x = a$, $y = b$ are maintained at temperature zero, and the plane $y = 0$ at steady temperature $v = \phi(x)$, show that, at any point of the solid,

$$v = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi}{a}(b - y)}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \int_0^a \phi(z) \sin \frac{n\pi z}{a}\, dz.$$

7. **If in the solid of *ex.* 6 the faces $x = 0$, $x = a$, $y = 0$ are kept at temperature zero, and the face $y = b$ at steady temperature $v = f(x)$, show that, at any point of the solid**

$$v = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi y}{a}}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \int_0^a f(z) \sin \frac{n\pi z}{a}\, dz.$$

8. If in the solid of *ex.* 6 the faces $x = 0$, $x = a$ are kept at temperature zero, the plane $y = 0$ at steady temperature $v = \phi(x)$, and the plane $y = b$ at steady temperature $v = f(x)$, show that, at any point of the solid $v = v_1 + v_2$, where v_1 and v_2 are the values of v found in *exs.* 6 and 7.

9. If in the solid of *ex.* 6, all the faces are kept at temperature zero, and $v = f(x, y)$ initially, show that, when $t > 0$,

$$v = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\kappa\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\times \int_0^a \sin\frac{m\pi\xi}{a}\, d\xi \int_0^b \sin\frac{n\pi\eta}{b} f(\xi, \eta)\, d\eta.$$

CHAPTER III

TRANSVERSE VIBRATIONS OF STRETCHED STRINGS

§ 1. **The Differential Equation.** It was in connection with the discussion of the vibrations of a stretched string that the question of the expansion of an arbitrary function in a series of sines and cosines first arose, though it was not till Fourier took up the subject in connection with the conduction of heat that such an expansion was definitely shown to be possible.

The string is supposed to be of uniform line-density ρ, and to be stretched with a tension P. The co-ordinate axes are taken to be rectangular, with the x-axis along the equilibrium position of the string. It is assumed that, in any disturbance of the string from its equilibrium position, the square of the inclination of any part to its initial direction may be neglected; that is, if (x, y, z) are the co-ordinates of any point on the string, the squares and higher powers of* $\frac{\partial y}{\partial x}$ and $\frac{\partial z}{\partial x}$ are negligible. At any particular moment the string lies along a curve in space, and y and z may be regarded as functions of x, so that

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right\}},$$

or, since $\left(\frac{\partial y}{\partial x}\right)^2$ and $\left(\frac{\partial z}{\partial x}\right)^2$ are to be neglected, $\frac{ds}{dx} = 1$. Hence, if it be assumed that the string as a whole does not move parallel to the x-axis, the length of any part of it remains unaltered. It follows that the tension P may be regarded as a constant, and also that the particles of the string may be assumed to vibrate in planes parallel to the (y, z) plane.

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* These derivatives are written as partial derivatives because y and z are functions of t as well as of x.

Now, consider a small element of the string of length δs, with x and $x + \delta x$ as the x-co-ordinates of its end-points. The forces acting on the element are the tensions at its ends, and any impressed forces $Y\rho\delta s$ and $Z\rho\delta s$ parallel to the y and z-axes respectively. At the end-point whose abscissa is x, the y and z components of the tension are $-P\frac{\partial y}{\partial s}$ and $-P\frac{\partial z}{\partial s}$, or, since the squares of $\frac{\partial y}{\partial x}$ and $\frac{\partial z}{\partial x}$ are to be neglected, $-P\frac{\partial y}{\partial x}$ and $-P\frac{\partial z}{\partial x}$. Similarly, at the end-point whose abscissa is $x + \delta x$, the corresponding components are

$$P\left\{\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\delta x\right\} \text{ and } P\left\{\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2}\delta x\right\},$$

δx being taken so small that its square and higher powers may be neglected. Thus the y and z forces due to tension acting on the element are $P\frac{\partial^2 y}{\partial x^2}\delta x$ and $P\frac{\partial^2 z}{\partial x^2}\delta x$ respectively. Hence the equations of motion are

$$\rho\delta x\frac{\partial^2 y}{\partial t^2} = P\frac{\partial^2 y}{\partial x^2}\delta x + Y\rho\delta x$$

and

$$\rho\delta x\frac{\partial^2 z}{\partial t^2} = P\frac{\partial^2 z}{\partial x^2}\delta x + Z\rho\delta x,$$

or

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + Y \quad . \quad . \quad . \quad (1)$$

and

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} + Z,$$

where $c^2 = P/\rho$.

It is evident from these equations that the dependent variables y and z are completely independent of each other. In what follows it will be assumed that the vibrations take place in the (x, y) plane, so that equation (1) only requires to be considered. As a rule, the impressed force Y will be zero, and equation (1) becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad (2)$$

The vibrations of the string are then described as Free Vibrations, while, if Y does not vanish, they are said to be Forced.

Kinetic and Potential Energies. The kinetic energy of the string is

$$T = \tfrac{1}{2}\rho\int \dot{y}^2 dx, \quad . \quad . \quad . \quad . \quad (3)$$

where $\dot{y}$ denotes $\frac{\partial y}{\partial t}$. The potential energy may be obtained by either of the two following methods.

Firstly, suppose that the string is brought from its equilibrium position into the configuration under consideration by lateral pressure, the alteration being brought about by steps such that, if y is the final ordinate of any point, its ordinate at an intermediate position is ky, where $0 < k < 1$, and k is constant for all values of x. In this intermediate position the force which must be applied to any element δx to balance the tensions at its ends will be

$$-\frac{\partial}{\partial x}\left(\mathrm{P}k\frac{\partial y}{\partial s}\right)\delta x,$$

where δx is so small that its square may be neglected, and this, when small quantities of higher order are omitted, may be equated to $-\mathrm{P}ky''\delta x$, where y'' denotes $\frac{\partial^2 y}{\partial x^2}$. Thus, since the displacement when k is increased by δk is $y\delta k$, the work done in bringing the element into its final position is

$$-\mathrm{P}yy''\delta x\int_0^1 k\,dk = -\tfrac{1}{2}\mathrm{P}yy''\delta x,$$

and consequently the potential energy of the string is

$$V = -\tfrac{1}{2}\mathrm{P}\int yy''dx \quad . \quad . \quad . \quad (4)$$

The second method of obtaining V is to equate it to the work done in lengthening the string against the tension. For any element δx of the string this is equal to

$$\mathrm{P}\left(\frac{ds}{dx} - 1\right)\delta x = \mathrm{P}\left[\sqrt{\left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}} - 1\right]\delta x,$$

or, approximately, $\tfrac{1}{2}\mathrm{P}\left(\frac{\partial y}{\partial x}\right)^2\delta x$;

and therefore the potential energy is

$$\int \tfrac{1}{2} \mathrm{P} \left(\frac{\partial y}{\partial x}\right)^2 dx,$$

or, if terms of higher orders of smallness be omitted,

$$\mathrm{V} = \tfrac{1}{2} \mathrm{P} \int y'^2 dx \quad . \quad . \quad . \quad . \quad (5)$$

The equivalence of the formulæ (4) and (5) can be established by integrating by parts in (4); this expression then becomes

$$- \tfrac{1}{2} \mathrm{P}[yy'] + \tfrac{1}{2} \mathrm{P} \int y'^2 dx,$$

and, since y vanishes at the ends of the disturbed portion of the string, the value of the first term is zero.

§ 2. **Solutions of the Equation.** The general solution of equation (2) is

$$y = f(ct - x) + \mathrm{F}(ct + x), \quad . \quad . \quad (6)$$

where f and F are arbitrary functions.

In the case of the function $f(ct - x)$, when t is increased by τ and x by $c\tau$, the value of the function is unaltered. Thus the equation

$$y = f(ct - x) \quad . \quad . \quad . \quad . \quad (7)$$

represents a wave-form moving with velocity c towards the right. For any values of x and t, y is called the *amplitude* of the wave. Similarly

$$y = \mathrm{F}(ct + x) \quad . \quad . \quad . \quad . \quad (8)$$

can be shown to represent a wave-form moving with velocity c towards the left.

Unlimited String. If the string is unlimited in length in both directions, and if the initial conditions are

$$y = \phi(x), \quad \dot{y} = \psi(x), \quad t = 0,$$

for all values of x, where $\phi(x)$ and $\psi(x)$ are arbitrary given functions, it can easily be verified that (6) becomes

$$y = \tfrac{1}{2}\{\phi(x - ct) + \phi(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi) d\xi. \quad (9)$$

Semi-infinite String. If the string is fixed at the origin, and extends to infinity in the direction of x positive, then, when x is zero, y vanishes for all values of t. Thus, from (6),

$$f(ct) + \mathrm{F}(ct) = 0,$$

for all values of t; so that, if z is any variable quantity,

$$F(z) = -f(z).$$

Hence the solution is of the form

$$y = f(ct - x) - f(ct + x), \quad . \quad . \quad (10)$$

and consequently involves only one arbitrary function.

Finite String Fixed at Both Ends. Let the string be fixed at $x = 0$ and $x = l$; then, when $x = l$, from (10),

$$0 = f(ct - l) - f(ct + l),$$

for all values of t; so that, if z be written for $ct - l$,

$$f(z) = f(z + 2l). \quad . \quad . \quad . \quad (11)$$

Thus $f(z)$ is a periodic function of period $2l$, and consequently the solution is of the form (10), subject to this proviso. The solution is then periodic also in t, of period $2l/c$: this period, it may be noted, is the time that a wave would take to travel twice the length of the string; so that a disturbance passing a point on the string will, after two successive reflections at the ends of the string, pass the point again in the same direction and with the same amplitude and sign.

It is with the vibrations of finite strings fixed at both ends that we are concerned in the theory of sound. In stringed musical instruments the strings are stretched with considerable tension between two points which limit the range of vibration. At one at least of these points the string passes over a bridge which rests on a sounding-board. The vibrations of the string are communicated, by means of the bridge, to the sounding-board, and from it to the surrounding air, the direct effects of the vibrations of the string on the air being negligible. The string may be caused to vibrate by three different methods: by plucking, as is done in playing the harp, by bowing, as in the case of the violin or violoncello, and by striking with a hammer, as in the case of the piano. Each of these cases will be considered separately.

§ 3. **Normal Modes of a Finite String.** A second method of solving equation (2) is as follows. Let $y = \text{JX}$, where J and X are respectively functions of t and x alone; then, substituting in (2) and dividing by c^2JX, we find that

$$\frac{\frac{d^2J}{dt^2}}{c^2J} = \frac{\frac{d^2X}{dx^2}}{X} \quad . \quad . \quad . \quad . \quad (12)$$

Now the function on the left of this equation is independent of x, and, in consequence, so is also the function on the right. It must therefore have a constant value, K say, so that

$$\frac{d^2X}{dx^2} = KX \quad . \quad . \quad . \quad . \quad (13)$$

The solution of this equation is

$$X = Ae^{x\sqrt{K}} + Be^{-x\sqrt{K}},$$

where A and B are arbitrary constants. But, when x is zero, y vanishes for all values of t, and, in consequence, X must also vanish. Thus $A + B = 0$, so that

$$X = A(e^{x\sqrt{K}} - e^{-x\sqrt{K}}).$$

But, when $x = l$, $X = 0$: hence $\sqrt{K}$ must be imaginary and equal to $in\pi/l$, where n is an integer: therefore

$$X = A2i \sin\frac{n\pi x}{l}.$$

Now, from (12) and (13), with this value of K,

$$\frac{d^2J}{dt^2} = -\frac{n^2\pi^2c^2}{l^2}J;$$

so that
$$J = C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l},$$

where C_n and D_n are arbitrary constants: hence one solution of (2) is

$$y = \left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}. \quad . \quad (14)$$

Since

$$2 \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} = \sin\frac{n\pi}{l}(ct + x) - \sin\frac{n\pi}{l}(ct - x),$$

and $$2 \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} = \cos\frac{n\pi}{l}(ct - x) - \cos\frac{n\pi}{l}(ct + x),$$

we see that the solution (14) is of the type given by (10) and (11). By expanding the latter solution in a Fourier sine series it can be deduced that it can be expressed in the form

$$y = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \quad . \quad (15)$$

which is the most general solution obtained from (14). If the string starts from rest, since $\dot{y} = 0$ initially, every D_n must vanish. If, on the other hand, it starts from the equilibrium position with given velocities, every C_n must vanish.

The solution (14) may also be written

$$y = K_n \sin \frac{n\pi x}{l} \cos \left(\frac{n\pi ct}{l} + E_n \right), \quad . \quad . \quad (16)$$

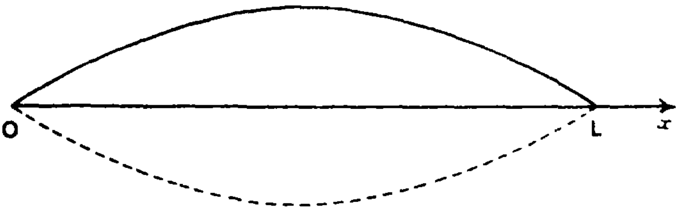

FIG. 5.

where $K_n^2 = C_n^2 + D_n^2$ and $\tan E_n = -D_n/C_n$. Here y varies as a simple harmonic function of the time, and the mode of vibration is called a *Normal Mode.* When the string vibrates in a normal mode, it gives a simple musical note, and the notes usually given by the string are combinations of the notes given by its normal modes of vibration.

The gravest, or fundamental mode, which determines the pitch of the note sounded, corresponds to $n = 1$. For this mode, at any time t the graph of y is in the form of a half-period sine curve, and it oscillates between two extreme positions such as those shown in Fig. 5. The frequency of this mode is

$$\frac{c}{2l} = \frac{1}{2l}\sqrt{\left(\frac{P}{\rho}\right)}, \quad . \quad . \quad . \quad (17)$$

and consequently varies inversely as the length, inversely as the square root of the density, and directly as the square root of the tension. The frequency of a note determines its pitch, the height of the note increasing with its frequency. Thus the note can be raised by tightening the string, and so increasing the tension, as is done in tuning the piano or the violin. It can also be raised by shortening the length; this the violin-player, in forming different notes on the same string, brings about by pressing with the tips of his fingers on different points of the string. The effect of line-density on pitch is also shown in the violin, in which the lower strings have the greater density. In the piano, in order to avoid having the lower strings of too great length, the density effect is increased by winding wires round the strings.

If, in (16), $n = 2$, the graph of y at any time is a complete sine-curve, and oscillates between extreme positions such as those shown in Fig. 6. The middle point $x = \frac{1}{2}l$ is always at rest, and this point is called a *Node*, the end-points also being nodes. For the nth mode there are $n - 1$ nodes, as well as the two end nodes. Midway between the nodes are the points of maximum amplitude, known as *Loops*. When the string is vibrating in the nth mode, each part of the string between two nodes is vibrating in the fundamental mode of a string of length l/n.

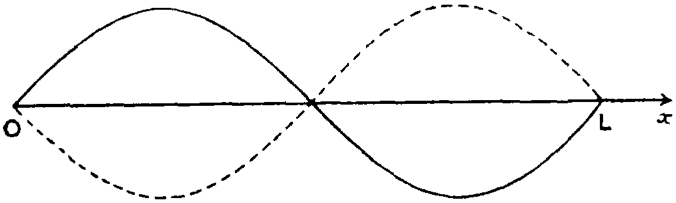

FIG. 6.

The note given by the fundamental mode is called the *Fundamental Tone*, and those given by the other modes are known as *Overtones* or *Harmonics*, the harmonic of order n corresponding to the nth mode. The harmonic of order 2 is an octave higher than the fundamental tone: those of orders 3, 4, 5, 6 also harmonise with it; while the harmonics of higher orders are usually so faint that they have no perceptible effect on the note sounded.

The kinetic energy of the normal mode (14) is

$$T = \tfrac{1}{2}\rho\int_0^l \dot{y}^2 dx = \frac{n^2\pi^2c^2}{4l}\rho\left(-C_n\sin\frac{n\pi ct}{l} + D_n\cos\frac{n\pi ct}{l}\right)^2, \quad (18)$$

while the potential energy is

$$V = \tfrac{1}{2}P\int_0^l y'^2 dx = \frac{n^2\pi^2}{4l}P\left(C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right)^2; \quad (19)$$

and therefore, since $c^2 = P/\rho$, the total energy is

$$T + V = \frac{n^2\pi^2}{4l}P(C_n^2 + D_n^2). \quad . \quad . \quad (20)$$

Similarly, from (15) it follows that the whole energy of the string is

$$T + V = \frac{\pi^2}{4l}P\sum_{n=1}^{\infty} n^2(C_n^2 + D_n^2), \quad . \quad . \quad (21)$$

so that the energy of the string is the sum of the energies of the normal modes. The loudness of any tone or overtone depends on the energy of the corresponding mode.

§ 4. **Vibrations of a Harp String.** In the case of the harp and other instruments, the string is set in vibration by plucking. If it is pulled through a distance b at a point distant a from the origin, and then released, its initial form will consist of the two straight lines OB and BL (Fig. 7), whose equations are

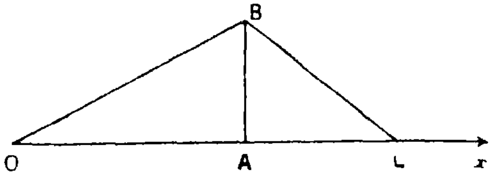

FIG. 7.

$$\left.\begin{aligned} y &= \frac{b}{a}x, \text{ for } 0 \leqq x \leqq a, \\ \text{and} \qquad y &= \frac{b}{l-a}(l-x), \text{ for } a \leqq x \leqq l \end{aligned}\right\} \quad . \quad . \quad (22)$$

Now, from (15), since the string is initially at rest,

$$y = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l},$$

and, when $t = 0$,

$$y = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}.$$

Hence, from (22),

$$C_n = \frac{2}{l}\int_0^a \frac{b}{a}x\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_a^l \frac{b}{l-a}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2bl^2}{n^2\pi^2a(l-a)}\sin\frac{n\pi a}{l}.$$

Thus, finally,

$$y = \frac{2bl^2}{\pi^2a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \qquad (23)$$

where $0 \leqq x \leqq l$, $0 \leqq t$.

It will be noticed that the harmonic of order n will be absent if $\sin\frac{n\pi a}{l} = 0$; that is, if the string is plucked at one of its nodes. For instance, if the string is plucked at its mid-point, all the harmonics of even order will be absent.

Since (23) can be written in the form

$$y = \frac{bl^2}{\pi^2a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi}{l}(x-ct)$$

$$+ \frac{bl^2}{\pi^2a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi}{l}(x+ct),$$

it is clear that the motion of the string is the resultant of two equal waves, moving in opposite directions with velocity c. Initially each wave is given by

$$y = \frac{bl^2}{\pi^2a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}, \qquad (24)$$

so that its ordinate at any point is half of that given by (22). Also, since (24) is an odd function of x, its graph for values of

x outside the range $0 \leq x \leq l$ is of the form shown in Fig. 8, where AC = $\frac{1}{2}$AB and L′ is the point $(-l, 0)$. If now this figure be moved from left to right with velocity c, and a figure initially coincident with it from right to left with the same velocity, and if the ordinates of these figures at time t be added, the resulting figure will give the position of the string at time t.

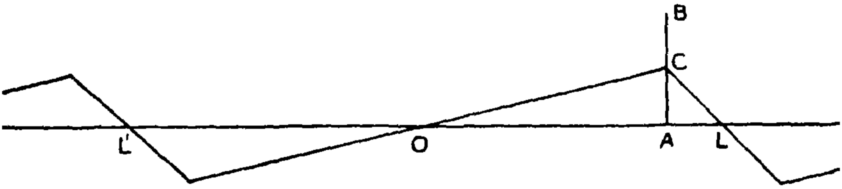

FIG. 8.

It is easily seen that the form of the string at any moment will consist of three portions, one a straight line parallel to OB, one a straight line parallel to BL, and between these a straight line whose gradient is the mean of the gradients of OB and BL. If the parallelogram OBLB′ (Fig. 9) be completed, the figure of the string at any time is obtained by moving a line PQ

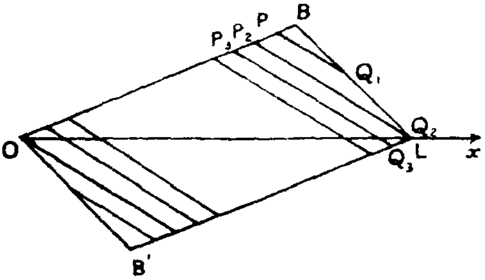

FIG. 9.

whose gradient is the mean of the gradients of OB and BL parallel to itself with uniform velocity from B to B′, and then back from B′ to B, the whole time taken for the double movement being the period of the fundamental mode of the string, namely, $2l/c$. Then, at any moment, the position of the string is that of OPQL. Any particle of the string moves backwards and forwards with uniform velocity between two extreme positions, in which, as a rule, it remains fixed for certain intervals.

From (23) and (21) it is clear that, apart from a trigonometrical factor which lies between 0 and 1, the intensities of the various harmonics will vary as $1/n^2$, so that the effect of the higher harmonics will be negligible.

§ 5. **Vibrations of a Violin String.** When a violin is being played, the frictional force exerted by the bow apparently drags the string with it until the string suddenly slips back; after brief intervals this process is repeated, the cycle taking place in the period of vibration of the string. The bow in this way supplies the string with energy to compensate for the loss of energy by dissipation, but the vibrations seem to be of the nature of free vibrations. The motion of any point of the string has been investigated experimentally, and it has been found that the point, for a time, moves with constant velocity, and then suddenly changes to a second constant velocity, which continues during the remainder of the period of vibration. Since the acceleration of the point is in general zero, it follows from (2) that the curvature of the string is zero; thus the form of the string at any instant is composed of straight lines, and all the conditions of the problem are satisfied if the string is in the form OBL (Fig. 10), where B is a variable point. Let B be the point (α, β); then the equations of the lines OB and BL are

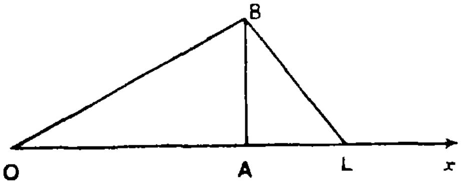

FIG. 10.

$$y_1 = \beta x/\alpha, \quad y_2 = \beta(l - x)\,(l - \alpha),$$

and the difference of the velocities on the two sides of B is

$$\dot{y}_1 - \dot{y}_2 = -\frac{l\beta\dot{\alpha}}{\alpha(l - \alpha)}, \quad . \quad . \quad . \quad (25)$$

since the coefficients of $\dot{\beta}$ cancel

In a short interval of time δt a length $\dot{\alpha}\delta t$ of the string is passed over by the moving point B, so that a mass $\rho\dot{\alpha}\delta t$ of the string has its velocity increased by the amount (25). This increase in momentum is caused by the transverse force

$$Py_2' - Py_1' = -\frac{Pl\beta}{\alpha(l-\alpha)}$$

acting for the time δt. Hence

$$-\frac{l\beta\dot{\alpha}}{\alpha(l-\alpha)}\rho\dot{\alpha}\delta t = -\frac{Pl\beta}{\alpha(l-\alpha)}\delta t$$

or

$$\dot{\alpha}^2 = P/\rho = c^2 \quad . \quad . \quad . \quad . \quad (26)$$

Thus the point B travels from left to right or from right to left with constant velocity c.

Let it be assumed that B starts from O, and moves from left to right with β positive. Then the particle whose abscissa is x moves, for a time x/c, with constant velocity

$$\dot{y}_2 = (l-x)\frac{d}{dt}\left(\frac{\beta}{l-\alpha}\right);$$

or, since $\dot{\alpha} = c$,

$$\dot{y}_2 = (l-x)c\frac{d}{d\alpha}\left(\frac{\beta}{l-\alpha}\right).$$

Now, since $\dot{y}_2$ and $l - x$ are constant for the particle,

$$\frac{d}{d\alpha}\left(\frac{\beta}{l-\alpha}\right) = C,$$

where C is a constant; therefore

$$\beta = C\alpha(l-\alpha) + D.$$

But, since β vanishes with α, D is zero; hence

$$\beta = C\alpha(l-\alpha) \quad . \quad . \quad . \quad . \quad (27)$$

It follows that B lies on an arc of a parabola passing through O and L, and with its axis bisecting OL at right angles. When β is negative, B lies on a similar arc below OL. Thus, in the period of vibration, B moves (Fig. 11) along the upper arc from O to L, and then along the lower arc from L to O, the velocity of A being constant and equal to $\pm c$. If the maximum displacement of B is β_0, then $C = 4\beta_0/l^2$, and the equations of the arcs are

$$\beta = \frac{4\beta_0}{l^2}\alpha(l - \alpha) \text{ and } \beta = -\frac{4\beta_0}{l^2}\alpha(l - \alpha) \quad . \quad (28)$$

The equations of OB and BL are then, when B lies on the upper arc,

$$y_1 = \frac{4\beta_0}{l^2}(l - \alpha)x \text{ and } y_2 = \frac{4\beta_0}{l^2}\alpha(l - x).$$

Initially, the string is in its equilibrium position, so that $y = 0$, and

$$\dot{y} = \dot{y}_2 = \frac{4\beta_0}{l^2}c(l - x). \quad . \quad . \quad . \quad (29)$$

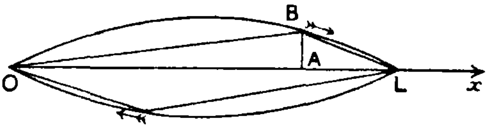

FIG. 11.

Hence, from (15),

$$y = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l},$$

and

$$\dot{y} = \sum_{n=1}^{\infty} D_n\frac{n\pi c}{l} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l},$$

where, from (29), when $t = 0$,

$$D_n\frac{n\pi c}{l} = \frac{2}{l}\int_0^l \frac{4\beta_0}{l^2}c(l - x)\sin\frac{n\pi x}{l}\,dx$$
$$= \frac{8\beta_0 c}{ln\pi}.$$

Thus

$$y = \frac{8\beta_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} \quad . \quad . \quad (30)$$

§ 6. **Vibrations of a Piano String.** If it is assumed that the blow of the hammer on the piano string is so sharp that the duration of the impact is small compared with the period of vibration of the string, the problem may be treated as one in which the string vibrates freely with the equilibrium position as initial position, and a given initial velocity spread over a small length of the string. Let σ be the momentum communicated to the string by the hammer, and a the abscissa of the point of impact: then, from (15),

$$y = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l},$$

and, initially, $\dot{y} = 0$, except for the points of the interval $(a - \epsilon, a + \epsilon)$ struck by the hammer, in which

$$\dot{y} = \sum_{n=1}^{\infty} D_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l},$$

where $\sigma = 2\epsilon\rho\dot{y}$. Thus

$$\begin{aligned} D_n\frac{n\pi c}{l} &= \frac{2}{l}\int_{a-\epsilon}^{a+\epsilon} \frac{\sigma}{2\epsilon\rho} \sin\frac{n\pi x}{l}\, dx \\ &= \frac{2\sigma}{n\pi\epsilon\rho} \sin\frac{n\pi\epsilon}{l} \sin\frac{n\pi a}{l}, \end{aligned}$$

and, when ϵ tends to zero, this tends to

$$\frac{2\sigma}{l\rho} \sin\frac{n\pi a}{l}.$$

Accordingly

$$y = \frac{2\sigma}{\pi\rho c} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}. \qquad (31)$$

In this solution the energies of the various modes are of the same order of magnitude, which, of course, is not in agreement with what actually happens. As a matter of fact, the impact is by no means instantaneous, but lasts for an interval of time which, though short, is comparable with the period of vibration of the string. In order to ensure this, the hammers are covered with cloth, so that the blow may not be too abrupt.

The vibration is therefore not a free vibration, but a forced vibration, and the differential equation is the equation (1), namely,

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + Y. \qquad (32)$$

It is not easy to find a suitable expression for Y, but a fair approximation is given by the formula

$$Y = \frac{1}{\pi}\frac{\mu\tau}{t^2 + \tau^2}, \quad . \quad . \quad . \quad (33)$$

where μ represents the time integral of the force Y from $t = -\infty$ to $t = +\infty$. The graph of Y is shown in Fig. 12. The formula has the disadvantage that there is no definite instant at which the impact begins or ends; but, by making τ small enough, the effect of the earlier and later parts of the blow can be made arbitrarily small, so that the formula will represent the actual conditions fairly well.

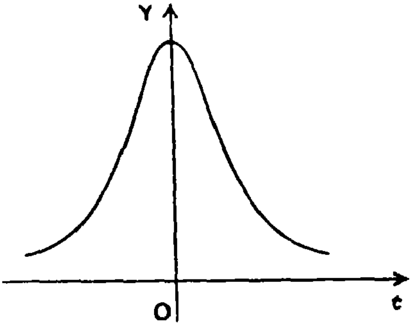

FIG. 12.

If the blow is spread over the interval $(a - \epsilon, a + \epsilon)$,

$$\mu = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}, \quad . \quad . \quad . \quad (34)$$

where

$$\begin{aligned} C_n &= \frac{2}{l}\int_{a-\epsilon}^{a+\epsilon} \mu \sin\frac{n\pi x}{l}\,dx \\ &= \frac{4\mu\epsilon\rho}{\rho n\pi\epsilon}\sin\frac{n\pi a}{l}\sin\frac{n\pi\epsilon}{l} \\ &= \frac{2\sigma}{\rho n\pi\epsilon}\sin\frac{n\pi a}{l}\sin\frac{n\pi\epsilon}{l}, \quad . \quad . \quad . \quad (35) \end{aligned}$$

σ being the total impulse communicated by the hammer to the string. When n is not large, and ϵ is small, the value of this is approximately

$$C_n = \frac{2\sigma}{\rho l}\sin\frac{n\pi a}{l}: \quad . \quad . \quad . \quad (36)$$

but when n tends to infinity, this is not the case, no matter how small ϵ may be; so that the series (34) converges, though very slowly.

Now, assume that,

$$y = \sum_{n=1}^{\infty} \eta_n \sin\frac{n\pi x}{l}, \quad . \quad . \quad . \quad (37)$$

where η_n is a function of the time: then, from (32), (33), and (34)

$$\frac{d^2\eta_n}{dt^2} + \frac{n^2\pi^2c^2}{l^2}\eta_n = \frac{1}{\pi}\frac{C_n\tau}{t^2+\tau^2}, \quad . \quad . \quad . \quad (38)$$

The solution of this equation is

$$\eta_n = \frac{lC_n\tau}{n\pi^2c}\left\{\sin\frac{n\pi ct}{l}\int^t \cos\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2} - \cos\frac{n\pi ct}{l}\int^t \sin\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2}\right\}, \quad (39)$$

where it is unnecessary to add explicitly terms of the type $A\cos\frac{n\pi ct}{l} + B\sin\frac{n\pi ct}{l}$, since the lower limits of the integrals in (39) are arbitrary. But, since the string was originally at rest in the position of equilibrium, we may write

$$\eta_n = \frac{lC_n\tau}{n\pi^2c}\left\{\sin\frac{n\pi ct}{l}\int_{-\infty}^t \cos\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2} - \cos\frac{n\pi ct}{l}\int_{-\infty}^t \sin\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2}\right\},$$

as this makes η_n and $\dot{\eta}_n$ both vanish when $t = -\infty$. Hence the value of η_n, after the impressed force has ceased to be sensible, is given by

$$\eta_n = A\cos\frac{n\pi ct}{l} + B\sin\frac{n\pi ct}{l},$$

where $$A = -\frac{lC_n\tau}{n\pi^2c}\int_{-\infty}^{\infty}\sin\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2} = 0,$$

and* $$B = \frac{lC_n\tau}{n\pi^2c}\int_{-\infty}^{\infty}\cos\frac{n\pi ct}{l}\frac{dt}{t^2+\tau^2} = \frac{lC_n}{n\pi c}e^{-\frac{n\pi c\tau}{l}}.$$

Thus $$\eta_n = \frac{lC_n}{n\pi c}e^{-\frac{n\pi c\tau}{l}}\sin\frac{n\pi ct}{l},$$

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* Gibson's Calculus, p. 469.

and, from (35) and (37),

$$y = \frac{2l\sigma}{\rho\pi^2 c\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi c\tau}{l}}\sin\frac{n\pi a}{l}\sin\frac{n\pi\epsilon}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \qquad (40)$$

or, from (36), if only the harmonics of comparatively small order are taken into account,

$$y = \frac{2\sigma}{\rho\pi c}\sum\frac{1}{n}e^{-\frac{n\pi c\tau}{l}}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} \quad . \qquad (41)$$

§ 7. **Vibrations of Membranes.** Consider a thin, tightly-stretched membrane, of uniform surface-density ρ, which, in its undisturbed position, lies in one plane. It is assumed that, if an imaginary line be drawn across the membrane in any direction, the mutual stresses between the two parts separated by the line are perpendicular to the line; and it can be shown that this stress is proportional to the length of the line and is independent of the direction of the line. The stress per unit length is called the "tension" of the membrane, and is denoted by P. It is also assumed that P is so great that it remains unaltered during the vibrations of the membrane, and that the square of the inclination of any line tangential to the surface of the membrane to the plane of equilibrium may be neglected.

Let the plane of equilibrium be the (x, y) plane; then the z-co-ordinate of any particle of the membrane will be the displacement of the particle. If now L, M, N, R be particles of the membrane whose projections on the (x, y) plane are the points (x, y), $(x + \delta x, y)$, $(x + \delta x, y + \delta y)$, $(x, y + \delta y)$, respectively, the tension across LR is $-\mathrm{P}\delta y$, and the z-component of this is $-\mathrm{P}\frac{\partial z}{\partial x}\delta y$, small quantities of higher order being neglected. Similarly the z-component of the tension across MN is

$$\mathrm{P}\left(\frac{\partial z}{\partial x}\right)_{x+\delta x}\delta y = \mathrm{P}\left(\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2}\delta x\right)\delta y,$$

and so the sum of these forces on the rectangular area LMNR is

$$\mathrm{P}\frac{\partial^2 z}{\partial x^2}\delta x\delta y.$$

In the same way it can be shown that the sum of the z-components of the tensions on LM and NR is

$$P\frac{\partial^2 z}{\partial y^2}\delta x\delta y.$$

Hence $$\rho\delta x\delta y\frac{\partial^2 z}{\partial t^2} = P\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\delta x\delta y,$$

or $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \quad . \quad . \quad . \quad (42)$$

where $c^2 = P/\rho$.

It may be shown, as in § 1, that the kinetic energy of the membrane is

$$T = \tfrac{1}{2}\rho\iint \dot{z}^2\,dxdy, \quad . \quad . \quad . \quad (43)$$

while the potential energy is

$$V = \tfrac{1}{2}P\iint\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}dxdy. \quad . \quad . \quad (44)$$

Rectangular Membrane. Consider now a membrane with a fixed boundary along the lines $x = 0$, $x = a$, $y = 0$, $y = b$ in the plane of equilibrium. Put $z = TXY$ in (42), where T, X, and Y are respectively functions of t, x, and y alone: then, in the same manner as in Chapter II, § 12, it can be shown that

$$\frac{d^2T}{dt^2} + p^2T = 0, \quad \frac{d^2X}{dx^2} + q^2X = 0, \quad \frac{d^2Y}{dy^2} + r^2Y = 0,$$

where $$p^2 = c^2(q^2 + r^2).$$

Also, as in that section, since $z = 0$ and consequently $X = 0$ when $x = 0$ and $x = a$, $q = m\pi/a$ and $X = \sin(m\pi x/a)$, where m is an integer. Similarly $r = n\pi/b$ and $Y = \sin(n\pi y/b)$, where n is an integer. Hence one solution of (42) is

$$z = (A\cos pt + B\sin pt)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad . \quad (45)$$

where $$p^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \quad . \quad . \quad . \quad (46)$$

and the most general solution obtained in this way is

$$z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{m,n}\cos pt + B_{m,n}\sin pt)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad (47)$$

Equation (45) gives a normal mode of vibration for the membrane. If the membrane starts from rest, $\dot{z}$ is initially zero, and therefore B vanishes. If, moreover, the initial displacement of the membrane is $z = f(x, y)$, then, from (47),

$$z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{A}_{m,n} \cos pt \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad . \quad (48)$$

where p is given by (46), and, when $t = 0$,

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{A}_{m,n} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

so that

$$\mathrm{A}_{m,n} = \frac{4}{ab}\int_0^a\int_0^b f(\xi, \eta) \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\eta d\xi. \quad . \quad (49)$$

Instead of nodal points we have, for the mode (45), *Nodal Lines* at

$$x = \frac{a}{m}, \frac{2a}{m}, \ldots, \frac{(m-1)a}{m}, \; y = \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{(n-1)b}{n}.$$

For combinations of modes the nodal lines are not necessarily straight lines. For instance, in the case of a square membrane, with $b = a$, the vibrations given by

$$z = \cos(pt + \epsilon)\left(\sin\frac{2\pi x}{a} \sin\frac{\pi y}{a} + \lambda \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a}\right)$$

$$= 2\cos(pt + \epsilon) \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}\left(\cos\frac{\pi x}{a} + \lambda\cos\frac{\pi y}{a}\right)$$

have the curved line

$$\cos\frac{\pi x}{a} + \lambda\cos\frac{\pi y}{a} = 0$$

as a nodal line.

CHAPTER IV

SPHERICAL HARMONICS: THE HYPERGEOMETRIC FUNCTION

§ 1. **The Potential Function.** Laplace, in a memoir written in 1782 and published in 1785, showed that the gravitational force at a point P due to a set of particles of masses $\mu_1, \mu_2, \mu_3, \ldots$ at the points $M_1, M_2, M_3, \ldots$ respectively can be obtained * by differentiating the function

$$V = \frac{\mu_1}{PM_1} + \frac{\mu_2}{PM_2} + \frac{\mu_3}{PM_3} + \ldots$$

At a later period this function was named by Green the *Potential* of the system at the point P, and this nomenclature is now universally employed.

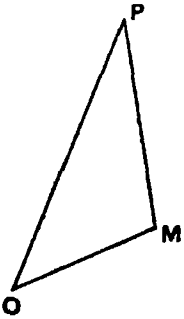

FIG. 13.

§ 2. **The Legendre Coefficients.** Legendre, to whom Laplace had communicated his potential theorem, investigated the expansion of a single term of the potential in the form of an infinite series, and was thus led (1782, or earlier), to the discovery of the functions now known as Legendre Coefficients.

Let a point O (Fig. 13) be taken as origin, and denote OP by r, OM by r_1, $\angle$ POM by θ, and cos θ by μ; then

$$\frac{1}{PM} = \frac{1}{\sqrt{(r^2 - 2rr_1\mu + r_1^2)}}.$$

If $r < r_1$,† this can be expanded in ascending powers of r in the form

$$\frac{1}{\sqrt{(r^2 - 2rr_1\mu + r_1^2)}} = P_0(\mu)\frac{1}{r_1} + P_1(\mu)\frac{r}{r_1^2} + P_2(\mu)\frac{r^2}{r_1^3} + \ldots, \quad (1)$$

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* Cf. Ch. VIII, § 2.

† Cf. Ch. V, § 1.

where the coefficients $P_0(\mu)$, $P_1(\mu)$, $P_2(\mu)$ . . . are polynomials in μ, known as the *Legendre Coefficients.* For example,

$$P_0(\mu) = 1,\ P_1(\mu) = \mu,\ P_2(\mu) = \frac{3\mu^2 - 1}{2},\ P_3(\mu) = \frac{5\mu^3 - 3\mu}{2}.$$

If $r > r_1$, $1/PM$ can be expanded in the series

$$\frac{1}{\sqrt{(r^2 - 2rr_1\mu + r_1^2)}} = P_0(\mu)\frac{1}{r} + P_1(\mu)\frac{r_1}{r^2} + P_2(\mu)\frac{r_1^2}{r^3} + \ . \ . \ . \quad (2)$$

The coefficient $P_n(\mu)$ is called the *Legendre Coefficient of degree* n or the *Legendre Polynomial of degree* n. As will be shown later, the Legendre coefficients are particular cases of Surface Spherical Harmonics.

§ 3. **Laplace's Coefficients.** Laplace, in the memoir mentioned above, besides introducing the idea of the potential, discussed the spherical harmonics from a more general point of view than that of Legendre, with whose work he was already acquainted.

In Fig. 13 let O be the origin, and let the rectangular co-ordinates of P and M be (x, y, z) and (x_1, y_1, z_1). If (r, θ, ϕ) and (r_1, θ_1, ϕ_1) are the corresponding polar co-ordinates,

$$\begin{aligned} x &= r \sin\theta \cos\phi, & x_1 &= r_1 \sin\theta_1 \cos\phi_1, \\ y &= r \sin\theta \sin\phi, & y_1 &= r_1 \sin\theta_1 \sin\phi_1, \\ z &= r \cos\theta, & z_1 &= r_1 \cos\theta_1, \end{aligned}$$

and, if γ be the angle POM,

$$\cos\gamma = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1). \quad (3)$$

With this value of $\cos\gamma$, $1/PM$ can be expanded in the forms

$$\frac{1}{\sqrt{(r^2 - 2rr_1\cos\gamma + r_1^2)}} = \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{r^n}{r_1^{n+1}},\ \text{if } r < r_1, \quad (4)$$

$$= \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{r_1^n}{r^{n+1}},\ \text{if } r > r_1. \quad (5)$$

The function $P_n(\cos\gamma)$ is a function of the two variables θ and ϕ, and is called the *Laplace's Coefficient of the* n*th degree.* When $\theta_1 = 0$ it degenerates into the corresponding Legendre Coefficient $P_n(\cos\theta)$.

§ 4. **Spherical Harmonics.** Thomson and Tait, in their well-known *Natural Philosophy* (1879) defined the spherical harmonics as follows. Any solution V_n of Laplace's Equation

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad . \quad (6)$$

which is homogeneous, of degree n, in x, y, z is called a *Solid Spherical Harmonic of degree n.* The degree n may be any number, and the function need not be rational.

Example. Verify that r^{-1}, 1, $ax + by + cz$, $x^2 - y^2 + yz$, $(z + ix)^n$ are solid spherical harmonics of degrees -1, 0, 1, 2, n respectively.

Kelvin's Theorem. If V_n is a solid spherical harmonic of degree n, $r^{-2n-1}V_n$ is a solid spherical harmonic of degree $-n-1$.

In (6) put $V = r^m V_n$; then

$$\frac{\partial V}{\partial x} = r^m \frac{\partial V_n}{\partial x} + mr^{m-2}xV_n,$$

$$\frac{\partial^2 V}{\partial x^2} = r^m \frac{\partial^2 V_n}{\partial x^2} + 2mr^{m-2}x\frac{\partial V_n}{\partial x} + m(m-2)r^{m-4}x^2 V_n + mr^{m-2}V_n,$$

and therefore, since, by Euler's Theorem,

$$x\frac{\partial V_n}{\partial x} + y\frac{\partial V_n}{\partial y} + z\frac{\partial V_n}{\partial z} = nV_n,$$

$$\begin{aligned} \nabla^2(r^m V_n) &= r^m \nabla^2 V_n + m(m + 2n + 1)r^{m-2}V_n, \\ &= m(m + 2n + 1)r^{m-2}V_n, \end{aligned}$$

since V_n satisfies Laplace's Equation. Hence, in order that $r^m V_n$ may satisfy Laplace's Equation, m must be zero or $-2n-1$. Thus $r^{-2n-1}V_n$ is a solid harmonic.

Example. 1 and $1/r$ are solid harmonics of degrees 0 and -1, x and x/r^3 of degrees 1 and -2.

Laplace's Equation, when expressed in terms of the cylindrical co-ordinates (u, ϕ, z) by means of the equations $x = u \cos \phi$, $y = u \sin \phi$ becomes

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad . \quad (7)$$

The further transformation $z = r\cos\theta$, $u = r\sin\theta$ to polar co-ordinates (r, θ, ϕ) changes it to the form

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0. \quad (8)$$

If this equation be multiplied by r^2, it can be written

$$r\frac{\partial^2(rV)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0. \qquad (9)$$

Now let $V_n = r^n U_n$ be a solution of this equation, where U_n is a function of θ and ϕ alone; V_n is, of course, a solid spherical harmonic of degree n. If $r^n U_n$ be substituted for V in (9), and the factor r^n removed, the equation becomes

$$n(n+1)U_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U_n}{\partial\phi^2} = 0, \qquad (10)$$

or, if μ be written for $\cos\theta$,

$$n(n+1)U_n + \frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial U_n}{\partial\mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 U_n}{\partial\phi^2} = 0. \qquad (11)$$

Surface Spherical Harmonics. The function U_n obtained by dividing V_n by r^n is called a *Surface Spherical Harmonic of degree* n. It is equal to V_n when $r = 1$, and is therefore given by the value of the corresponding solid spherical harmonic on the surface of the sphere whose centre is the origin and whose radius is of unit length (the unit sphere). The necessary and sufficient condition that a function U_n of θ and ϕ should be a surface spherical harmonic of degree n is that it should satisfy equation (10).

It will now be shown that the Laplace's Coefficient of degree n is a surface spherical harmonic of degree n. By differentiating the function

$$T = \frac{1}{\sqrt{\{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\}}}$$

with regard to x, y, and z it can be verified that T satisfies Laplace's Equation. Thus if T be expressed in the form

$$\frac{1}{\sqrt{(r^2 - 2rr_1\cos\gamma + r_1^2)}} = \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{r^n}{r_1^{n+1}}$$

and substituted in (9) it is found that

$$\sum_{n=0}^{\infty}\frac{r^n}{r_1^{n+1}}\left[n(n+1)P_n(\cos\gamma) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial P_n(\cos\gamma)}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2 P_n(\cos\gamma)}{\partial\phi^2}\right] = 0$$

As this equation holds for all values of r less than r_1, the coefficients of the different powers of r all vanish identically. Accordingly

$$n(n+1)P_n(\cos\gamma) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial P_n(\cos\gamma)}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2 P_n(\cos\gamma)}{\partial\phi^2} = 0,$$

so that $P_n(\cos\gamma)$ satisfies (10), and is therefore a surface spherical harmonic.

§ 5. **Legendre's Equation.** The Legendre coefficient $P_n(\cos\theta)$, being a particular case of $P_n(\cos\gamma)$ (obtained by putting $\theta_1 = 0$), is a surface spherical harmonic of degree n which is independent of ϕ. It therefore satisfies the equation

$$n(n+1)y + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right) = 0,$$

so that, if μ be written in place of $\cos\theta$, $P_n(\mu)$ satisfies the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dy}{d\mu}\right\} + n(n+1)y = 0,$$

or

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + n(n+1)y = 0. \quad . \quad (12)$$

This is known as *Legendre's Equation.*

§ 6. **Legendre's Associated Equation.** If in equation (11) we put $U_n = \Theta\Phi$, where Θ and Φ are functions of θ and ϕ alone, and divide by $\Theta\Phi$, it becomes

$$n(n+1) + \frac{1}{\Theta}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0.$$

Now the first two terms in this equation are independent of ϕ, and therefore so is also the last. Hence the value of $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ must be constant. Since the value of Φ when ϕ is increased by 2π is usually unaltered, it is convenient to take this constant to be $-m^2$, where m is usually an integer. Thus

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi$$

and

$$\Phi = A\cos m\phi + B\sin m\phi,$$

where A and B are arbitrary constants. Then Θ satisfies the equation

$$\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{dy}{d\mu}\right\} + \left\{n(n + 1) - \frac{m^2}{1 - \mu^2}\right\}y = 0, \qquad (13)$$

which is known as Legendre's Associated Equation. If Θ is a solution of this equation, any function of the form

$$(A \cos m\phi + B \sin m\phi)\Theta$$

satisfies (11) and is therefore a surface spherical harmonic of degree n, while

$$r^n (A \cos m\phi + B \sin m\phi)\Theta$$

and $$r^{-n-1}(A \cos m\phi + B \sin m\phi)\Theta$$

are solid spherical harmonics of degrees n and $-n-1$ respectively.

§ 7. **The Hypergeometric Function.** As the Hypergeometric Function is frequently employed in connection with the theory of the spherical harmonics, an account of some of its properties will be given here.

The function is defined by means of the *Hypergeometric Series*—

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)}x^2$$
$$+ \frac{\alpha(\alpha + 1)(\alpha + 2)\beta(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma + 1)(\gamma + 2)}x^3 + \ldots \qquad (14)$$

which is absolutely convergent if $|x| < 1$. If $|x| = 1$, it converges absolutely if $\gamma - \alpha - \beta > 0$.

Particular cases of the hypergeometric series are:

$$(1 + x)^n = F(-n, 1, 1, -x), \qquad (15)$$
$$\log(1 + x) = xF(1, 1, 2, -x), \qquad (16)$$
$$\sin^{-1}x = xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x^2), \qquad (17)$$
$$\tan^{-1}x = xF(\tfrac{1}{2}, 1, \tfrac{3}{2}, -x^2). \qquad (18)$$

§ 8. **The Hypergeometric Equation.** The differential equation

$$x(1 - x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0 \qquad (19)$$

is known as *Gauss's Equation* or the *Hypergeometric Equation.* To find a solution in the form of an infinite series of ascending powers of x we put

$$y = x^\rho(c_0 + c_1x + c_2x^2 + \ldots) \qquad (20)$$

and substitute for y in (19): the left-hand side of (19) is then equal to

$$(1-x)x^{\rho-1}\sum_{n=0}^{\infty}(\rho+n)(\rho+n-1)c_n x^n$$

$$+\{\gamma-(\alpha+\beta+1)x\}x^{\rho-1}\sum_{n=0}^{\infty}(\rho+n)c_n x^n-\alpha\beta x^{\rho}\sum_{n=0}^{\infty}c_n x^n,$$

or

$$c_0\{\rho(\rho-1)+\gamma\rho\}x^{\rho-1}$$

$$+\sum_{n=0}^{\infty}\begin{bmatrix}c_{n+1}\{(\rho+n+1)(\rho+n)+\gamma(\rho+n+1)\}\\ -c_n\{(\rho+n)(\rho+n-1)+(\rho+n)(\alpha+\beta+1)+\alpha\beta\}\end{bmatrix}x^{\rho+n}.$$

In order, then, that y should satisfy (19), the coefficients of all the powers of x in this series must vanish. Now c_0, being the coefficient of the first term, cannot be zero, and therefore, in order that the coefficient of $x^{\rho-1}$ may vanish, ρ must satisfy the equation

$$\rho(\rho-1+\gamma)=0. \quad . \quad . \quad . \quad (21)$$

As the values of the index ρ are obtained from this equation, it is called the *Indicial Equation*: its roots are $\rho = 0$ and $\rho = 1 - \gamma$.

From the coefficients of the other terms we obtain the equations

$$c_{n+1}(\rho+n+1)(\rho+n+\gamma)=c_n(\rho+n+\alpha)(\rho+n+\beta), \quad (22)$$

where $n = 0, 1, 2, 3, \ldots$; and by means of these equations the coefficients can all be expressed in terms of c_0. Taking the values 0 and $1-\gamma$ of ρ we obtain the two solutions

$$y_1 = c_0\mathrm{F}(\alpha, \beta, \gamma, x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$y_2 = c_0 x^{1-\gamma}\mathrm{F}(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x), \quad (24)$$

unless γ is an integer. If $\gamma = 1$ the two solutions are identical, and if γ tends to any other integral value one of the integrals usually ceases to exist.

Example 1. Show that

$$\sin nx = n\sin x\,\mathrm{F}\left(\frac{1+n}{2}, \frac{1-n}{2}, \tfrac{3}{2}, \sin^2 x\right), \quad . \quad . \quad (25)$$

and

$$\cos nx = \mathrm{F}\left(\frac{n}{2}, -\frac{n}{2}, \tfrac{1}{2}, \sin^2 x\right). \quad . \quad . \quad . \quad . \quad (26)$$

The functions $\sin nx$, $\cos nx$ satisfy the equation

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

If this equation is transformed by means of the substitution $u = \sin^2 x$, it becomes

$$u(1 - u)\frac{d^2y}{du^2} + (\tfrac{1}{2} - u)\frac{dy}{du} + \frac{n^2}{4}y = 0,$$

which is Gauss's Equation with $\alpha = \frac{1}{2}n$, $\beta = -\frac{1}{2}n$, $\gamma = \frac{1}{2}$. Thus the general solution is

$$y = \mathrm{A\,F}\left(\frac{n}{2}, -\frac{n}{2}, \tfrac{1}{2}, \sin^2 x\right) + \mathrm{B}\sin x\,\mathrm{F}\left(\frac{1+n}{2}, \frac{1-n}{2}, \tfrac{3}{2}, \sin^2 x\right).$$

But $\sin nx$ and $\cos nx$ are solutions : if $y = \sin nx$, A must be zero, since y vanishes with x : hence

$$\frac{\sin nx}{\sin x} = \mathrm{B\,F}\left(\frac{1+n}{2}, \frac{1-n}{2}, \tfrac{3}{2}, \sin^2 x\right).$$

On making x tend to zero, this gives $n = \mathrm{B}$, from which (25) follows. Again, if $y = \cos nx$, by putting $x = 0$ we see that $\mathrm{A} = 1$: also, on differentiating, and putting $x = 0$, we find that $\mathrm{B} = 0$. Thus we deduce formula (26).

Example 2. Show that

(i) $\sin nx = n \sin x \cos x\ \mathrm{F}\left(1 + \frac{n}{2}, 1 - \frac{n}{2}, \tfrac{3}{2}, \sin^2 x\right)$,

(ii) $\cos nx = \cos x\ \mathrm{F}\left(\frac{1+n}{2}, \frac{1-n}{2}, \tfrac{1}{2}, \sin^2 x\right)$.

§ 9. The Four Forms of the Hypergeometric Function.

In the integral

$$\mathrm{I} = \int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1}(1 - xt)^{-\alpha}dt,$$

where $\beta > 0$, $\gamma - \beta > 0$, assume that $|x| < 1$, and expand by the binomial theorem ; then, integrating term by term, we get

$$\mathrm{I} = \sum_{r=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+r-1)}{r!}x^r \int_0^1 (1 - t)^{\gamma-\beta-1}t^{\beta+r-1}dt$$

$$= \sum_{r=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+r-1)}{r!}x^r\mathrm{B}(\beta + r, \gamma - \beta),$$

and therefore

$$\int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1}(1 - xt)^{-\alpha}dt = \mathrm{B}(\beta, \gamma - \beta)\mathrm{F}(\alpha, \beta, \gamma, x). \tag{27}$$

This equation can be used to define the Hypergeometric Function for values of x which do not satisfy the inequality $|x| < 1$.

Again, if in I we replace t by $1 - t$, we find that

$$\begin{aligned} \mathrm{I} &= \int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1}(1-x+xt)^{-\alpha}dt \\ &= (1-x)^{-\alpha}\int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1}\left(1-\frac{x}{x-1}t\right)^{-\alpha}dt \\ &= \mathrm{B}(\beta, \gamma-\beta)(1-x)^{-\alpha}\mathrm{F}\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right). \end{aligned}$$

Thus

$$\mathrm{F}(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha}\,\mathrm{F}\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right).$$

In this equation interchange α and β, then

$$\mathrm{F}(\alpha, \beta, \gamma, x) = (1-x)^{-\beta}\,\mathrm{F}\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right),$$

and therefore

$$\mathrm{F}\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) = (1-x)^{\alpha-\beta}\mathrm{F}\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right).$$

Here replace β by $\gamma - \beta'$, and x by $\frac{x'}{x'-1}$, and the equation becomes

$$\mathrm{F}(\alpha, \beta', \gamma, x') = (1-x')^{\gamma-\alpha-\beta}\,\mathrm{F}(\gamma-\beta', \gamma-\alpha, \gamma, x').$$

Hence

$$\begin{aligned} \mathrm{F}(\alpha, \beta, \gamma, x) &= (1-x)^{\gamma-\alpha-\beta}\mathrm{F}(\gamma-\alpha, \gamma-\beta, \gamma, x) \quad (28) \\ &= (1-x)^{-\alpha}\mathrm{F}\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \quad (29) \\ &= (1-x)^{-\beta}\mathrm{F}\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right), \quad (30) \end{aligned}$$

and these are the four forms of the hypergeometric function.

§ 10. **The Asymptotic Expansion of the Hypergeometric Function.** A formula for the remainder in the binomial expansion can be obtained as follows. We have

$$\int_0^1 (1+zt)^{m-1}dt = \{(1+z)^m - 1\}/(mz),$$

and therefore

$$\begin{aligned}(1 + z)^m &= 1 + mz\int_0^1 (1 + zt)^{m-1}dt \\ &= 1 + mz\Big[-(1 - t)(1 + zt)^{m-1}\Big]_0^1 \\ &\quad + \frac{m(m-1)}{2\,!}z^2\int_0^1 2(1 - t)(1 + zt)^{m-2}dt \\ &= 1 + mz + \frac{m(m-1)}{2\,!}z^2\Big[-(1 - t)^2(1 + zt)^{m-2}\Big]_0^1 \\ &\quad + \frac{m(m-1)(m-2)}{3\,!}z^3\int_0^1 3(1 - t)^2(1 + zt)^{m-3}dt,\end{aligned}$$

and so on. Proceeding thus, we find that

$$(1 + z)^m = \sum_{r=0}^{s-1} \frac{\Gamma(m+1)}{r\,!\,\Gamma(m - r + 1)}z^r + R'_s, \quad . \quad . \quad . \quad (31)$$

where $$R'_s = \frac{\Gamma(m+1)}{s\,!\,\Gamma(m - s + 1)}z^s\int_0^1 s(1 - t)^{s-1}(1 + zt)^{m-s}dt, \quad (32)$$

provided that $(1 + zt)$ does not vanish for $0 \leqq t \leqq 1$.

Now apply this formula to the binomial expression $(1 - xt)^{-\alpha}$ in (27), in which it is assumed that x is not real and greater than or equal to 1, then

$$\begin{aligned}B(\beta, \gamma - \beta)F(\alpha, \beta, \gamma, x) &= \sum_{r=0}^{s-1}\left\{\frac{\Gamma(-\alpha + 1)}{r\,!\,\Gamma(-\alpha - r + 1)}(-x)^r \right. \\ &\quad \left. \times \int_0^1 t^{\beta + r - 1}(1 - t)^{\gamma - \beta - 1}dt\right\} + \text{Remainder} \\ &= \sum_{r=0}^{s-1}\frac{\Gamma(-\alpha + 1)}{r\,!\,\Gamma(-\alpha - r + 1)}B(\beta + r, \gamma - \beta)(-x)^r + \text{Remainder};\end{aligned}$$

and so, on division by $B(\beta, \gamma - \beta)$, we find that

$F(\alpha, \beta, \gamma, x) =$ *the first* s *terms in the series for* $F(\alpha, \beta, \gamma, x) + R_s$, (33)

$$\text{where } R_s = T_{s+1}\frac{\displaystyle\int_0^1 s(1 - \lambda)^{s-1}d\lambda\int_0^1 \frac{t^{\beta + s - 1}(1 - t)^{\gamma - \beta - 1}}{(1 - \lambda t x)^{\alpha + s}}dt}{B(\beta + s, \gamma - \beta)},$$

T_{s+1} being the $(s + 1)^{\text{th}}$ term in the series for $F(\alpha, \beta, \gamma, x)$.

If x is not real and greater than or equal to 1, $|(1 - \lambda t x)| > 0$; let M_s be the greatest value of $|(1 - \lambda t x)^{-\alpha - s}|$ for $0 \leqq \lambda \leqq 1$, $0 \leqq t \leqq 1$; then

$$|R_s| \leqq |T_{s+1}| \times M_s \frac{\int_0^1 s(1-\lambda)^{s-1}d\lambda \int_0^1 t^{\beta+s-1}(1-t)^{\gamma-\beta-1}dt}{B(\beta+s, \gamma-\beta)}$$

$$= |T_{s+1}| \times M_s$$

Now, if γ is positive, $|T_{s+1}|$ can be made as small as we please by increasing γ, and this is true even when $|x|$ is not less than 1. Hence, if x is not real and greater than 1, $|R_s|$ can be made as small as we please by increasing γ, and therefore the first s terms in (33) will give an approximate value of the function. An expansion of this kind, consisting of a finite number of terms and a remainder which can be made arbitrarily small by sufficiently increasing the variable, is called an *Asymptotic Expansion*. Thus the series for $F(\alpha, \beta, \gamma, x)$ is asymptotic in γ, provided that $\gamma > 0$, and that x is not real and greater than 1.

§ 11. **Formulae for the Gamma Function and other Functions.** The following formulae are given here for convenience in reference :—

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!\, n^x}{x(x+1)\ldots(x+n)} \qquad (34)$$

$$\frac{1}{\Gamma(x)} = e^{\gamma x} x \prod_{n=1}^{\infty} \left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}\right\}, \qquad (35)$$

where γ is Euler's Constant: its value is approximately 0·57722.

$$\Gamma(x+1) = x\Gamma(x). \qquad (36)$$

If n is a positive integer, $\Gamma(n+1) = n!$ (37)

$$\Gamma(1) = 1. \qquad (38)$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad (39)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \qquad (40)$$

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\tfrac{1}{2})2^{2x-1} \qquad (41)$$

If $x > 0$, $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$. (42)

Stirling's Formula is

$$\lim_{n\to\infty} \Gamma(n+1)\Big/\left\{\sqrt{(2\pi n)}\left(\frac{n}{e}\right)^n\right\} = 1 \quad . \quad (43)$$

If $0 < x < 1$,

$$\int_0^\infty \cos t \,.\, t^{x-1}dt = \Gamma(x)\cos\left(\tfrac{1}{2}\pi x\right) \quad . \quad . \quad (44)$$

The Beta Function is $B(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ . . . (45)

If $p > 0$, $q > 0$, $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}dx = B(p, q)$. (46)

If $m > 0$, $n > 0$, $\displaystyle 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta = B(m, n)$ (47)

$$\psi(x) = \frac{d}{dx}\log\Gamma(x+1) \quad . \quad . \quad . \quad (48)$$

If n is a positive integer

$$\psi(x+n) = \psi(x) + \sum_{r=1}^{n}\frac{1}{x+r} \quad . \quad . \quad (49)$$

If n is a positive integer

$$\psi(n) = \phi(n) - \gamma, \quad . \quad . \quad . \quad . \quad (50)$$

where $\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. (51)

$$\psi(0) = -\gamma \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\psi(-x-1) = \psi(x) + \pi\cot\pi x \quad . \quad . \quad . \quad (53)$$

$$2\psi(2x) = \psi(x) + \psi\left(x - \tfrac{1}{2}\right) + 2\log 2 \quad . \quad (54)$$

$$\psi\left(-\tfrac{1}{2}\right) = -\gamma - 2\log 2. \quad . \quad . \quad . \quad (55)$$

Gauss's Theorem for the Hypergeometric Function is

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad . \quad . \quad (56)$$

provided that $\gamma > 0$, $\gamma - \alpha - \beta > 0$.

Examples.

1. Prove that the function $f(ax + by + cz)$ satisfies Laplace's Equation, provided that $a^2 + b^2 + c^2 = 0$.
2. Show that $(z + ix\cos a + iy\sin a)^n$ is a solid harmonic of degree n, and deduce that $\{\cos\theta + i\sin\theta\cos(\phi - a)\}^n$ is a surface harmonic of degree n.

3. If V is a solution of Laplace's equation, show that $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)V$ is also a solution, $f(p, q, r)$ being a rational integral function of p, q, r.

$$\left[\nabla^2 f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)V = f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\nabla^2 V\right].$$

4. If $F(x, y, z)$ is any function of x, y, z which satisfies Laplace's Equation, show that the function $\frac{1}{r}F\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$ also satisfies Laplace's Equation.

5. Prove that the most general surface spherical harmonic of degree zero can be expressed in any of the three forms:

(i) $V = f(\chi + i\phi) + F(\chi - i\phi),$

where $\chi = \log \tan \frac{1}{2}\theta$ and f and F are arbitrary functions;

(ii) $V = \Phi(e^{i\phi} \tan \frac{1}{2}\theta) + \Psi(e^{-i\phi} \tan \frac{1}{2}\theta),$

where Φ and Ψ are arbitrary functions;

(iii) $V = \Phi\left(\frac{x + iy}{r + z}\right) + \Psi\left(\frac{x - iy}{r + z}\right).$

[Show that, when $n = 0$, (10) can be written $\frac{\partial^2 V}{\partial \chi^2} + \frac{\partial^2 V}{\partial \phi^2} = 0.$

The most general solution of this equation is (i).]

CHAPTER V

THE LEGENDRE POLYNOMIALS

§ 1. **Legendre's Expansion.** Consider that branch of the function $(1 - 2\mu z + z^2)^{-\frac{1}{2}}$ which has the value + 1 when $z = 0$. This function of z is discontinuous when $1 - 2\mu z + z^2 = 0$; *i.e.*, when $z = \mu \pm \sqrt{(\mu^2 - 1)}$. For values of z such that $|z|$ is less than the smaller of the two quantities $|\mu \pm \sqrt{(\mu^2 - 1)}|$ the function can, by Taylor's Theorem, be expanded in ascending powers of z in the form (Ch. IV, § 2),

$$P_0(\mu) + zP_1(\mu) + z^2P_2(\mu) + z^3P_3(\mu) + \ldots \quad (1)$$

If $-1 \leqq \mu \leqq 1$; *i.e.*, if $\mu = \cos\theta$, where θ is real,

$$\mu \pm \sqrt{(\mu^2 - 1)} = \cos\theta \pm i\sin\theta,$$

so that $|\mu \pm \sqrt{(\mu^2 - 1)}| = 1$ and the expansion is valid for $|z| < 1$.

By expanding both sides of the equation

$$\{1 - 2(-\mu)z + z^2\}^{-\frac{1}{2}} = \{1 - 2\mu(-z) + (-z)^2\}^{-\frac{1}{2}}$$

in powers of z, and equating the coefficients of corresponding powers of z, it can be seen that

$$P_n(-\mu) = (-1)^n P_n(\mu); \quad (2)$$

thus the polynomial $P_n(\mu)$ is even if n is even and odd if n is odd.

Again, if $\mu = 1$,

$$(1 - 2z + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(1)z^n = (1 - z)^{-1} = \sum_{n=0}^{\infty} z^n,$$

so that

$$P_n(1) = 1. \quad (3)$$

From (2) it follows that

$$P_n(-1) = (-1)^n. \quad (4)$$

§ 2. **Expansion of $P_n(\mu)$ in Powers of μ.** For values of μ and z such that $|2\mu z - z^2| < 1$, $(1 - 2\mu z + z^2)^{-\frac{1}{2}}$ can be expanded by the binomial theorem in the form

$$1 + \tfrac{1}{2}z(2\mu - z) + \frac{1 \cdot 3}{2 \cdot 4}z^2(2\mu - z)^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^3(2\mu - z)^3 + \quad . \; . \; . \quad (5)$$

If, moreover, μ and z are such that $2|\mu z| + |z^2| < 1$, the terms of this series can be rearranged in any way without altering the value of the series; and, for any value of μ, $|z|$ may be chosen so small that this inequality is satisfied. Hence, equating the coefficients of z^n in (1) and (5), commencing, in the latter case, with the $(n + 1)^{\text{th}}$ term of the series and considering the terms of the series in their reverse order, we have

$$\begin{aligned} P_n(\mu) &= \frac{1 \cdot 3 \; . \; . \; . \; (2n-1)}{2 \cdot 4 \; . \; . \; . \; . \; (2n)}\left\{(2\mu)^n - \frac{2n}{2n-1}\,\frac{n-1}{1}(2\mu)^{n-2}\right. \\ &\qquad \left. + \frac{2n(2n-2)}{(2n-1)(2n-3)}\,\frac{(n-2)(n-3)}{1 \cdot 2}(2\mu)^{n-4} - \; . \; . \; . \right\} \\ &= \frac{1 \cdot 3 \; . \; . \; . \; (2n-1)}{1 \cdot 2 \; . \; . \; . \; . \; n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2}\right. \\ &\qquad \left. + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)}\mu^{n-4} - \; . \; . \; . \right\} \qquad (6) \\ &= \frac{1 \cdot 3 \; . \; . \; . \; (2n-1)}{1 \cdot 2 \; . \; . \; . \; . \; n}\mu^n F\left(-\frac{n}{2}, \frac{1-n}{2}, \tfrac{1}{2} - n, \frac{1}{\mu^2}\right). \qquad (6') \end{aligned}$$

If n is even, the hypergeometric series in (6′) contains $\frac{1}{2}n + 1$ terms, the last being

$$(-1)^{\frac{1}{2}n}\frac{n(n-1)(n-2) \; . \; . \; . \; 1}{2 \cdot 4 \; . \; . \; . \; n(2n-1)(2n-3) \; . \; . \; . \; (n+1)}\,\frac{1}{\mu^n},$$

while if n is odd there are $\frac{1}{2}(n+1)$ terms, the last being

$$(-1)^{\frac{1}{2}(n-1)}\frac{n(n-1) \; . \; . \; . \; 3 \cdot 2}{2 \cdot 4 \; . \; . \; . \; (n-1)(2n-1)(2n-3) \; . \; . \; . \; (n+2)}\,\frac{1}{\mu^{n-1}}.$$

It is left to the reader to verify the following expansions:—

$P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$, $P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)$,
$P_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$, $P_5(\mu) = \frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)$,
$P_6(\mu) = \frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)$,
$P_7(\mu) = \frac{1}{16}(429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu)$,

$$P_8(\mu) = \tfrac{1}{128}(6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35),$$
$$P_9(\mu) = \tfrac{1}{128}(12155\mu^9 - 25740\mu^7 + 18018\mu^5 - 4620\mu^3 + 315\mu),$$
$$P_{10}(\mu) = \tfrac{1}{256}(46189\mu^{10} - 109395\mu^8 + 90090\mu^6 - 30030\mu^4 + 3465\mu^2 - 63).$$

Example. By differentiating (6) show that

$$\frac{d^m P_n(\mu)}{d\mu^m} = \frac{(2n)!}{2^n . n!\,(n-m)!}\left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 . 4(2n-1)(2n-3)}\mu^{n-m-4} - \dots\right\}$$

§ 3. **Expansion of $P_n(\mu)$ in Powers of $\frac{1}{2}(1-\mu)$.** The function $(1 - 2\mu z + z^2)^{-\frac{1}{2}}$ can be written

$$\{(1-z)^2 + 2z(1-\mu)\}^{-\frac{1}{2}} = \frac{1}{1-z}\left\{1 + \frac{4z}{(1-z)^2}\,\frac{1-\mu}{2}\right\}^{-\frac{1}{2}},$$

and, if z be taken sufficiently small, the latter expression can be expanded by the binomial theorem in the form

$$\frac{1}{1-z} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1-z}\,\frac{1 . 3 \dots (2n-1)}{2 . 4 \dots (2n)}\left\{\frac{4z}{(1-z)^2}\,\frac{1-\mu}{2}\right\}^n.$$

In this series expand each term in powers of z, and pick out the coefficients of z^n; thus

$$\begin{aligned} P_n(\mu) &= 1 - \frac{1}{2}\,\frac{n(n+1)}{2!}\left(4\,\frac{1-\mu}{2}\right) \\ &\qquad + \frac{1}{2}\,\frac{3}{4}\,\frac{(n-1)n(n+1)(n+2)}{4!}\left(4\,\frac{1-\mu}{2}\right)^2 + \dots \\ &= 1 + \frac{(n+1)(-n)}{1 . 1}\left(\frac{1-\mu}{2}\right) \\ &\qquad + \frac{(n+1)(n+2)(-n)(-n+1)}{1 . 2 . 1 . 2}\left(\frac{1-\mu}{2}\right)^2 + \dots \\ &= F\left(n+1,\ -n,\ 1,\ \frac{1-\mu}{2}\right). \qquad (7) \end{aligned}$$

Corollary. From (2) it follows that

$$P_n(\mu) = (-1)^n F\left(n+1,\ -n,\ 1,\ \frac{1+\mu}{2}\right) \qquad (8)$$

Again, by expanding the factors on both sides of the equation

$$(1 - 2\cos\theta . z + z^2)^{-\frac{1}{2}} = (1 - ze^{i\theta})^{-\frac{1}{2}}(1 - ze^{-i\theta})^{-\frac{1}{2}}$$

in powers of z, it can be shown that

$$\sum_{n=0}^{\infty} P_n(\cos\theta) \,.\, z^n = \left(1 + \tfrac{1}{2}ze^{i\theta} + \frac{1 \,.\, 3}{2 \,.\, 4}z^2e^{2i\theta} + \,.\,.\,.\,\right)$$

$$\times \left(1 + \tfrac{1}{2}ze^{-i\theta} + \frac{1 \,.\, 3}{2 \,.\, 4}z^2e^{-2i\theta} + \,.\,.\,.\,\right).$$

If now the coefficients of z^n are equated, it follows that

$$P_n(\cos\theta) = \frac{(2n)!}{2^{2n}(n\,!)^2}\left\{2\cos n\theta + \frac{1 \,.\, n}{1 \,.\, (2n-1)}2\cos(n-2)\theta\right.$$

$$\left. + \frac{1 \,.\, 3}{1 \,.\, 2}\,\frac{n(n-1)}{(2n-1)(2n-3)}2\cos(n-4)\theta + \,.\,.\,.\,\right\}, \quad (9)$$

the series ending, if n is odd, with the term which contains $2\cos\theta$ as a factor, while, if n is even, the corresponding factor of the last term is $\cos(0\,.\,\theta) = 1$.

From (9) it follows that $P_n(\cos\theta)$ has its greatest value when $\theta = 0$, as each term is then positive and has its maximum value. Thus, from (3) and (4)

$$-1 \leqq P_n(\cos\theta) \leqq 1. \quad . \quad . \quad . \quad (10)$$

§ 4. **Rodrigues' Formula.** From (6) it follows that

$$\int_0^{\mu} P_n(\mu)d\mu = \frac{1 \cdot 3 \cdot 5 \,.\,.\,.\, (2n-1)}{(n+1)!}\left\{\mu^{n+1} - \frac{(n+1)n}{2(2n-1)}\mu^{n-1}\right.$$

$$\left. + \frac{(n+1)n(n-1)(n-2)}{2 \,.\, 4(2n-1)(2n-3)}\mu^{n-3} - \,.\,.\,.\,\right\},$$

and, if this operation be repeated other $n - 1$ times, the resulting function is

$$\frac{1 \,.\, 3 \,.\,.\,.\, (2n-1)}{(2n)!}\left\{\mu^{2n} - \frac{2n(2n-1)}{2 \,.\, (2n-1)}\mu^{2n-2}\right.$$

$$\left. + \frac{2n(2n-1)(2n-2)(2n-3)}{2 \,.\, 4 \,.\, (2n-1)(2n-3)}\mu^{2n-4} - \,.\,.\,.\,\right\}$$

$$= \frac{1}{2 \,.\, 4 \,.\,.\,.\, (2n)}(\mu^2 - 1)^n.$$

The nth derivative of this function is therefore $P_n(\mu)$, so that

$$P_n(\mu) = \frac{1}{2^n \,.\, n\,!}\,\frac{d^n}{d\mu^n}(\mu^2 - 1)^n. \quad . \quad . \quad (11)$$

This is the well-known *Rodrigues' Formula.*

Jacobi's Proof of Rodrigues' Formula. The formula can also be established by means of Lagrange's Expansion. Let

$$\lambda = \mu + \tfrac{1}{2}\alpha(\lambda^2 - 1); \qquad . \quad . \quad . \quad (12)$$

then that value of λ which is equal to μ when $\alpha = 0$ is given by the Lagrange's Expansion—

$$\lambda = \mu + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{d\mu^{n-1}}\left(\frac{\mu^2 - 1}{2}\right)^n.$$

Therefore

$$\frac{d\lambda}{d\mu} = 1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^n}{d\mu^n}\left(\frac{\mu^2 - 1}{2}\right)^n.$$

Again, from (12),

$$\lambda = \alpha^{-1}\{1 - \sqrt{(1 - 2\mu\alpha + \alpha^2)}\},$$

so that

$$\frac{d\lambda}{d\mu} = \frac{1}{\sqrt{(1 - 2\mu\alpha + \alpha^2)}} = 1 + \sum_{n=1}^{\infty} \alpha^n P_n(\mu).$$

Hence, if the coefficients of α^n in the two expansions for $\frac{d\lambda}{d\mu}$ are equated, the formula is obtained.

Example 1. Prove that $P_n(\mu) = F\left(-n, n + 1, 1, \frac{1 - \mu}{2}\right)$.

[Put Rodrigues' Formula in the form

$$P_n(\mu) = \frac{(-1)^n}{n!} \frac{d^n}{d\mu^n}\left\{(1 - \mu)^n\left(1 - \frac{1 - \mu}{2}\right)^n\right\},$$

expand the last factor in powers of $(1 - \mu)\ 2$, and differentiate.]

Example 2. If m is a positive integer, show that

$$\frac{d^m}{d\mu^m} P_n(\mu) = \frac{(n + m)!}{2^m . m!(n - m)!} F\left(m - n, m + n + 1, m + 1, \frac{1 - \mu}{2}\right).$$

Rodrigues' General Formula. The formula

$$\frac{d^{n-m}(\mu^2 - 1)^n}{d\mu^{n-m}} = \frac{(n - m)!}{(n + m)!}(\mu^2 - 1)^m \frac{d^{n+m}(\mu^2 - 1)^n}{d\mu^{n+m}}, \qquad (13)$$

which will be found useful, can be established as follows.

Let $\mu^2 - 1 = u$, so that $(\mu + \lambda)^2 - 1 = u + 2\mu\lambda + \lambda^2$: then, expanding by Taylor's Theorem, we have

$$\{(\mu + \lambda)^2 - 1\}^n = u^n + \sum_{r=1}^{2n} \frac{\lambda^r}{r!} \frac{d^r u^n}{d\mu^r}.$$

Now divide this equation by λ^n, and rearrange the terms on the right; this gives

$$\left(\frac{u + 2\mu\lambda + \lambda^2}{\lambda}\right)^n = \frac{1}{n!}\frac{d^n u^n}{d\mu^n} + \sum_{m=1}^{n} \frac{\lambda^m}{(n+m)!}\frac{d^{n+m}u^n}{d\mu^{n+m}}$$
$$+ \sum_{m=1}^{n} \frac{\lambda^{-m}}{(n-m)!}\frac{d^{n-m}u^n}{d\mu^{n-m}}. \qquad (14)$$

Here replace λ by u/λ; the left-hand side remains unaltered, and the resulting equation is

$$\left(\frac{u + 2\mu\lambda + \lambda^2}{\lambda}\right)^n = \frac{1}{n!}\frac{d^n u^n}{d\mu^n} + \sum_{m=1}^{n} \frac{\lambda^{-m}u^m}{(n+m)!}\frac{d^{n+m}u^n}{d\mu^{n+m}}$$
$$+ \sum_{m=1}^{n} \frac{\lambda^m u^{-m}}{(n-m)!}\frac{d^{n-m}u^n}{d\mu^{n-m}}. \qquad (15)$$

Since the expressions on the right of (14) and (15) are identically equal, we may equate the coefficients of λ^{-m} and so obtain formula (13).

Legendre's Equation. By means of (11) and (13) it is easy to find the differential equation (IV, 12) which is satisfied by $P_n(\mu)$. From (11)

$$P_n(\mu) = c\frac{d^n}{d\mu^n}(\mu^2 - 1)^n,$$

where c is a constant. Hence, from (13), with $m = 1$,

$$(\mu^2 - 1)\frac{dP_n(\mu)}{d\mu} = n(n+1)c\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2 - 1)^n;$$

and so, on differentiating, we get

$$\frac{d}{d\mu}\left\{(\mu^2 - 1)\frac{dP_n(\mu)}{d\mu}\right\} = n(n+1)P_n(\mu).$$

Thus $P_n(\mu)$ satisfies the equation (IV, 12).

§ 5. **The Zeros of** $P_n(\mu)$. From Rodrigues' Formula it can be deduced that the n zeros of the polynomial $P_n(\mu)$ are all real and lie between -1 and $+1$.

The function $f(\mu) = (\mu^2 - 1)^n$ has n zeros at 1 and n zeros at -1. By Rolle's Theorem $f'(\mu)$ must have at least one zero between -1 and $+1$. But $f'(\mu)$ has $2n - 1$ zeros in all, of which $n - 1$ are at -1 and $n - 1$ at $+1$: thus there is only one zero between -1 and $+1$.

Similarly $f''(\mu)$ has $n - 2$ zeros at -1, $n - 2$ at $+1$, and 2 between -1 and $+1$. Proceeding in this way we finally deduce that $f^{(n)}(\mu)$ or $P_n(\mu)$ has n zeros between -1 and $+1$. [The values of $P_n(\mu)$ at -1 and $+1$ are $(-1)^n$ and 1 respectively.]

When n is even, $P_n(\mu)$ is an even function, so that the zeros occur in pairs, equal in magnitude but opposite in sign; when n is odd, $\mu = 0$ is a zero, and, since $P_n(\mu)/\mu$ is even, the others occur in equal and opposite pairs.

That all these zeros are distinct can be deduced from the following theorem :—

Theorem. If y is any solution of the linear differential equation

$$ay'' + by' + cy = 0, \qquad . \quad . \quad . \quad (16)$$

where x is the independent variable, and a, b, c are functions of x which are continuous and have all their derivatives continuous, the function y cannot have any repeated zeros except possibly for values of x which satisfy $a = 0$.

For if y has a repeated zero, then $y = 0$ and $y' = 0$; hence, from (16), since a is not zero, y'' must be zero. Now let (16) be differentiated, and it will be seen that $y''' = 0$; by proceeding in this way it can be shown that all the derivatives of y must vanish, and therefore, by Taylor's Theorem, y vanishes identically.

Thus $P_n(\mu)$, being a solution of Legendre's Equation (IV, 12), in which a is of the form $(1 - \mu^2)$, has no repeated zeros between -1 and $+1$; *i.e.*, all its zeros are distinct.

§ 6. **Integrals of Products of Legendre Polynomials.** The following theorems can be deduced from Rodrigues' Formula :—

Theorem I. If the positive integers m and n are unequal

$$\int_{-1}^{1} P_m(\mu)P_n(\mu)d\mu = 0. \qquad . \qquad . \qquad . \qquad (17)$$

Let m be greater than n, and denote the integral by I; then

$$I = \frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^m}{d\mu^m}(\mu^2 - 1)^m\frac{d^n}{d\mu^n}(\mu^2 - 1)^n d\mu.$$

Now integrate by parts; thus

$$I = \frac{1}{2^{m+n}m!\,n!}\left[\frac{d^{m-1}}{d\mu^{m-1}}(\mu^2 - 1)^m\frac{d^n}{d\mu^n}(\mu^2 - 1)^n\right]_{-1}^{1}$$
$$- \frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^{m-1}}{d\mu^{m-1}}(\mu^2 - 1)^m\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2 - 1)^n d\mu.$$

Since $(\mu^2 - 1)$ is a factor of $\frac{d^{m-1}}{d\mu^{m-1}}(\mu^2 - 1)^m$ the first term vanishes. By repeating this operation other $n - 1$ times it can be shown that

$$I = \frac{(-1)^n}{2^{m+n}.\,m!\,n!}\int_{-1}^{1}\frac{d^{m-n}}{d\mu^{m-n}}(\mu^2 - 1)^m\frac{d^{2n}}{d\mu^{2n}}(\mu^2 - 1)^n d\mu.$$

But $\frac{d^{2n}}{d\mu^{2n}}(\mu^2 - 1)^n = (2n)!$: hence

$$I = \frac{(-1)^n(2n)!}{2^{m+n}.\,m!\,n!}\left[\frac{d^{m-n-1}}{d\mu^{m-n-1}}(\mu^2 - 1)^m\right]_{-1}^{1} = 0.$$

Corollary. If $m + n$ is even and $m \neq n$, $\int_{0}^{1} P_m(\mu)P_n(\mu)d\mu = 0$.

Theorem II. $$\int_{-1}^{1}\{P_n(\mu)\}^2 d\mu = \frac{2}{2n + 1}. \qquad . \qquad . \qquad (18)$$

As in *Theorem I.* it is found that

$$\int_{-1}^{1}\{P_n(\mu)\}^2 d\mu = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^{n-n}}{d\mu^{n-n}}(\mu^2 - 1)^n d\mu$$
$$= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1 - \mu^2)^n d\mu.$$

Here put $\mu = 2x - 1$; then

$$\int_{-1}^{1}\{P_n(\mu)\}^2 d\mu = \frac{2\,.\,(2n)!}{(n!)^2}\int_0^1 x^n(1-x)^n dx$$

$$= \frac{2\,.\,(2n)!}{(n!)^2}B(n+1,\, n+1) = \frac{2}{2n+1}.$$

Corollary. $\displaystyle\int_0^1 \{P_n(\mu)\}^2 d\mu = \frac{1}{2n+1}.$

Theorem III. If $R(m)$[*] $> r - 1$, where r is a positive integer,

$$\int_0^1 \mu^m P_r(\mu)d\mu = \frac{m(m-1)(m-2)\ldots(m-r+2)}{(m+r+1)(m+r-1)\ldots(m-r+3)}. \qquad (19)$$

For, if I denote the integral, then, by Rodrigues' Theorem,

$$I = \int_0^1 \mu^m P_r(\mu)d\mu = \frac{1}{2^r\,.\,r!}\int_0^1 \mu^m \frac{d^r}{d\mu^r}(\mu^2-1)^r d\mu,$$

and, on integrating by parts r times, it is found that

$$I = (-1)^r\frac{m(m-1)\ldots(m-r+1)}{2^r\,.\,r!}\int_0^1 \mu^{m-r}(\mu^2-1)^r d\mu$$

$$= \frac{m(m-1)\ldots(m-r+1)}{2^r\,.\,r!}\int_0^1 \mu^{m-r}(1-\mu^2)^r d\mu$$

$$= \frac{m(m-1)\ldots(m-r+1)}{2^{r+1}\,.\,r!}\int_0^1 \lambda^{\frac{m-r-1}{2}}(1-\lambda)^r d\lambda, \text{ where } \lambda = \mu^2,$$

$$= \frac{m(m-1)\ldots(m-r+1)}{2^{r+1}\,.\,r!}B\left(\frac{m-r+1}{2},\, r+1\right)$$

$$= \frac{m(m-1)\ldots(m-r+1)}{2^{r+1}}\frac{\Gamma\left(\frac{m-r+1}{2}\right)}{\Gamma\left(\frac{m+r+3}{2}\right)}$$

$$= \frac{m(m-1)\ldots(m-r+1)}{(m+r+1)(m+r-1)\ldots(m-r+1)}$$

$$= \frac{m(m-1)\ldots(m-r+2)}{(m+r+1)(m+r-1)\ldots(m-r+3)}.$$

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[*] If z is a complex quantity, $R(z)$ denotes its real part and $I(z)$ its imaginary part. It can be shown by other methods that (19) and (20) hold for any values of m for which the integrals are convergent. [Cf. *ex.* 22, p. 107.]

Corollary. If $R(m) > r - 1$,

$$\int_0^1 \mu^m P_r(\mu)d\mu = \frac{m(m-2)\ldots(m-r+2)}{(m+r+1)(m+r-1)\ldots(m+1)}, \text{ if } r \text{ is even,}$$

$$= \frac{(m-1)(m-3)\ldots(m-r+2)}{(m+r+1)(m+r-1)\ldots(m+2)}, \text{ if } r \text{ is odd.} \quad (20)$$

Alternative Proof of Theorem I. The first of these theorems can also be proved as follows. The functions $P_n(\mu)$ and $P_m(\mu)$ satisfy the corresponding Legendre Equations :—

$$\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{dP_n}{d\mu}\right\} + n(n + 1)P_n = 0,^*$$

$$\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{dP_m}{d\mu}\right\} + m(m + 1)P_m = 0.$$

Multiply these equations by P_m and P_n respectively, subtract, and integrate; thus

$$\int_{-1}^1 P_m\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{dP_n}{d\mu}\right\}d\mu - \int_{-1}^1 P_n\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{dP_m}{d\mu}\right\}d\mu$$

$$+ [n(n + 1) - m(m + 1)]\int_{-1}^1 P_nP_m d\mu = 0.$$

Integration by parts in the first two terms of this equation gives

$$\int_{-1}^1 (1 - \mu^2)\frac{dP_m}{d\mu}\frac{dP_n}{d\mu}d\mu - \int_{-1}^1 (1 - \mu^2)\frac{dP_n}{d\mu}\frac{dP_m}{d\mu}d\mu = 0,$$

and therefore, since $[n(n+1) - m(m+1)] = (n - m)(n + m + 1)$ is not zero ($n \neq m$),

$$\int_{-1}^1 P_nP_m d\mu = 0.$$

Alternative Proof of Theorem II. Since the series (1) is absolutely convergent when $|z| < 1$, it is permissible to square it ; thus

$$(1 - 2\mu z + z^2)^{-1} = \sum_{n=0}^{\infty} P_n^2 z^{2n} + 2\sum P_nP_m z^{n+m},$$

where in the second summation m and n are unequal.

---

* When there is no dubiety regarding the argument of the function, we may write P_n in place of $P_n(\mu)$.

On integrating both sides of this equation, and applying *Theorem I.*, we get

$$\int_{-1}^{1} \frac{d\mu}{1 - 2\mu z + z^2} = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{1} P_n^2 d\mu. \qquad (21)$$

But the integral on the left has the value

$$\frac{1}{z} \log \left(\frac{1 + z}{1 - z} \right) = 2\left(\frac{1}{1} + \frac{z^2}{3} + \frac{z^4}{5} + \ldots \right); \qquad (22)$$

and therefore, on equating the coefficients of z^{2n} in (21) and (22), we find that

$$\int_{-1}^{1} P_n^2 \, d\mu = \frac{2}{2n + 1}.$$

§ 7. **Expression of a Polynomial as a Linear Function of Legendre Polynomials.** The polynomial

$$f(\mu) = A_0\mu^n + A_1\mu^{n-1} + \ldots + A_n$$

can be expressed in the form

$$B_nP_n(\mu) + B_{n-1}P_{n-1}(\mu) + \ldots + B_0P_0(\mu).$$

For, since the coefficient of μ^n in $P_n(\mu)$ is $(2n)!/\{2^n(2n!)^2\}$,

$$f(\mu) - A_0 \frac{2^n(n!)^2}{(2n)!} P_n(\mu)$$

is a polynomial, $\phi(\mu)$ say, of degree $n - 1$ at most: thus, if $B_n = A_0 2^n(n!)^2/(2n)!$,

$$f(\mu) = B_nP_n(\mu) + \phi(\mu).$$

Similarly $\phi(\mu) = B_{n-1}P_{n-1}(\mu) + \psi(\mu)$, where $\psi(\mu)$ is a polynomial of degree $n - 2$ at most; so that

$$f(\mu) = B_nP_n(\mu) + B_{n-1}P_{n-1}(\mu) + \psi(\mu).$$

Proceeding in this way we can express $f(\mu)$ in the required form.

Note. The value of any coefficient B_m can be found as follows. Multiply the equation

$$f(\mu) = B_nP_n(\mu) + B_{n-1}P_{n-1}(\mu) + \ldots + B_0P_0(\mu), \qquad (23)$$

by $P_m(\mu)$, and integrate from -1 to $+1$; then, from (17), the coefficients of all the B's except B_m will vanish, while, from

(18), the coefficient of B_m will have the value $2/(2m+1)$. Accordingly

$$B_m = \frac{2m+1}{2}\int_{-1}^{1} f(\mu)P_m(\mu)d\mu \quad . \quad . \quad (24)$$

Corollary. It follows that

$$\mu^n = A_nP_n(\mu) + A_{n-2}P_{n-2}(\mu) + A_{n-4}P_{n-4}(\mu) + \ldots,$$

where $A_n = 2^n(n!)^2/(2n)!$. The actual values of the coefficients can be obtained from (20); the resulting expressions are,

(i) when n is even,

$$\begin{aligned}\mu^n = {} & \frac{n(n-2)\ldots 2}{(2n+1)(2n-1)\ldots(n+1)}(2n+1)P_n(\mu) \\ & + \frac{n(n-2)\ldots 4}{(2n-1)(2n-3)\ldots(n+1)}(2n-3)P_{n-2}(\mu) \\ & + \ldots + \frac{1}{n+1}P_0(\mu), \quad . \quad . \quad . \quad . \quad (25)\end{aligned}$$

(ii) when n is odd

$$\begin{aligned}\mu^n = {} & \frac{(n-1)(n-3)\ldots 2}{(2n+1)(2n-1)\ldots(n+2)}(2n+1)P_n(\mu) \\ & + \frac{(n-1)(n-3)\ldots 4}{(2n-1)(2n-3)\ldots(n+2)}(2n-3)P_{n-2}(\mu) \\ & + \ldots + \frac{1}{n+2}3P_1(\mu) \quad . \quad . \quad . \quad . \quad (25')\end{aligned}$$

It is left to the reader to verify the following particular cases of these formulæ :—

$$\begin{aligned}
&\mu^0 = P_0,\ \mu^1 = P_1,\ \mu^2 = \tfrac{2}{3}P_2 + \tfrac{1}{3}P_0,\ \mu^3 = \tfrac{2}{5}P_3 + \tfrac{3}{5}P_1, \\
&\mu^4 = \tfrac{8}{35}P_4 + \tfrac{4}{7}P_2 + \tfrac{1}{5}P_0,\ \mu^5 = \tfrac{8}{63}P_5 + \tfrac{4}{9}P_3 + \tfrac{3}{7}P_1, \\
&\mu^6 = \tfrac{16}{231}P_6 + \tfrac{24}{77}P_4 + \tfrac{10}{21}P_2 + \tfrac{1}{7}P_0, \\
&\mu^7 = \tfrac{16}{429}P_7 + \tfrac{8}{39}P_5 + \tfrac{14}{33}P_3 + \tfrac{1}{3}P_1, \\
&\mu^8 = \tfrac{128}{6435}P_8 + \tfrac{64}{495}P_6 + \tfrac{48}{143}P_4 + \tfrac{40}{99}P_2 + \tfrac{1}{9}P_0, \\
&\mu^9 = \tfrac{128}{12155}P_9 + \tfrac{192}{2431}P_7 + \tfrac{16}{65}P_5 + \tfrac{56}{143}P_3 + \tfrac{3}{11}P_1, \\
&\mu^{10} = \tfrac{256}{46189}P_{10} + \tfrac{128}{2717}P_8 + \tfrac{32}{187}P_6 + \tfrac{48}{143}P_4 + \tfrac{50}{143}P_2 + \tfrac{1}{11}P_0.
\end{aligned}$$

Example. Show that, if $m < n$,

$$\text{(i)} \quad \int_{-1}^{1} \mu^m P_n(\mu)d\mu = 0,$$

$$\text{(ii)} \quad \int_{-1}^{1} \mu^n P_n(\mu)d\mu = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Expansion of $P'_n(\mu)$ *in a series of Legendre Polynomials.* Since $P'_n(\mu)$ is a polynomial of degree $n - 1$,

$$P'_n(\mu) = \Sigma B_m P_m(\mu),$$

where $m = n - 1, n - 3, n - 5, \ldots$ and

$$B_m = \frac{2m + 1}{2}\int_{-1}^{1} P'_n(\mu)P_m(\mu)d\mu.$$

Now integrate the expression on the right by parts; thus

$$B_m = \frac{2m + 1}{2}\Big[P_n(\mu)P_m(\mu)\Big]_{-1}^{1} - \frac{2m + 1}{2}\int_{-1}^{1} P_n(\mu)P'_m(\mu)d\mu.$$

But we can write

$$P'_m(\mu) = C_{m-1}P_{m-1}(\mu) + C_{m-3}P_{m-3}(\mu) + \ldots,$$

so that, as $n > m - 1$, the second integral on the right has the value zero. Accordingly

$$B_m = \frac{2m + 1}{2}[1 - (-1)^{m+n}] = 2m + 1,$$

since $m + n$ is odd. Therefore

$$P'_n(\mu) = (2n - 1)P_{n-1}(\mu) + (2n - 5)P_{n-3}(\mu) + (2n - 9)P_{n-5}(\mu) + \ldots \quad (26)$$

Corollary 1. $P''_n(\mu) = (2n - 3)(2n - 1 \cdot 1)P_{n-2}(\mu) + (2n - 7)(4n - 2 \cdot 3)P_{n-4}(\mu) + (2n - 11)(6n - 3 \cdot 5)P_{n-6}(\mu) + \quad (26')$

The proof of this formula is left as an exercise to the reader.

Corollary 2. From (10) and (26) it follows that, for $-1 \leqq \mu \leqq 1$,

$$| P'_{2n}(\mu) | \leqq n(2n + 1), \quad | P'_{2n+1}(\mu) | \leqq (n + 1)(2n + 1). \quad (27)$$

Expansion of $\mu P'_n(\mu)$ *in a series of Legendre Polynomials.* Again,

$$\mu P'_n(\mu) = \Sigma B_m P_m(\mu),$$

where $m = n, n - 2, n - 4, \ldots$ and

$$B_m = \frac{2m + 1}{2}\int_{-1}^{1} \mu P'_n(\mu)P_m(\mu)d\mu.$$

If the expression on the right be integrated by parts, it gives

$$B_m = \frac{2m+1}{2}\Big[\mu P_n(\mu)P_m(\mu)\Big]_{-1}^{1} - \frac{2m+1}{2}\int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu$$
$$- \frac{2m+1}{2}\int_{-1}^{1} P_n(\mu)\mu P'_m(\mu)d\mu.$$

Since $m + n$ is even, the first term has the value $2m + 1$. If $m < n$, the second term has the value zero, and, by expanding $\mu P'_m(\mu)$ as a series of Legendre Polynomials, it can be shown that the same is true of the third term. For $m = n$

$$B_n = 2n + 1 - 1 - B_n,$$

so that $B_n = n$ Hence, finally,

$$\mu P'_n(\mu) = nP_n(\mu) + (2n - 3)P_{n-2}(\mu)$$
$$+ (2n - 7)P_{n-4}(\mu) + . \qquad (28)$$

§ 8. **The Recurrence Formulæ.** From (26) we have

$$P'_{n+1}(\mu) = (2n+1)P_n(\mu) + (2n-3)P_{n-2}(\mu) + (2n-7)P_{n-4}(\mu) + \ldots$$
$$P'_{n-1}(\mu) = \qquad (2n-3)P_{n-2}(\mu) + (2n-7)P_{n-4}(\mu) + \ldots,$$

from which, by subtraction, it follows that, for $n = 1, 2, 3, \ldots$

$$P'_{n+1}(\mu) - P'_{n-1}(\mu) = (2n + 1)P_n(\mu) \quad . \qquad (29)$$

For $n = 0$ the corresponding formula is

$$P'_1(\mu) = P_0(\mu) \,. \quad . \quad . \qquad (30)$$

Again, from (28),

$$\mu P'_{n+1}(\mu) = (n+1)P_{n+1}(\mu) + (2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu) + \ldots$$
$$\mu P'_{n-1}(\mu) = \qquad (n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu) + \ldots,$$

and therefore, by subtraction,

$$\mu\{P'_{n+1}(\mu) - P'_{n-1}(\mu)\} = (n + 1)P_{n+1}(\mu) + nP_{n-1}(\mu).$$

From this formula and (29) it follows that, for $n = 1, 2, 3, \ldots,$

$$(n + 1)P_{n+1}(\mu) - (2n + 1)\mu P_n(\mu) + nP_{n-1}(\mu) = 0, \qquad (31)$$

while, for $n = 0$, the corresponding formula is

$$P_1(\mu) - \mu P_0(\mu) = 0. \quad . \quad . \quad . \qquad (32)$$

Alternative Proof. These formulæ may also be derived as follows. Differentiate

$$(1 - 2\mu z + z^2)^{-\frac{1}{2}} = \sum_{0}^{\infty} z^n P_n(\mu) \quad . \quad . \quad (33)$$

with regard to z; then

$$(\mu - z)(1 - 2\mu z + z^2)^{-\frac{3}{2}} = \sum_{1}^{\infty} n z^{n-1} P_n(\mu). \quad . \quad (34)$$

Now multiply both sides of this equation by $(1 - 2\mu z + z^2)$: thus

$$(\mu - z)\sum_{0}^{\infty} z^n P_n(\mu) = (1 - 2\mu z + z^2)\sum_{1}^{\infty} n z^{n-1} P_n(\mu),$$

from which, by equating the coefficients of z^n, it is found that

$$\mu P_n(\mu) - P_{n-1}(\mu) = (n + 1)P_{n+1}(\mu) - 2n\mu P_n(\mu) \\ + (n - 1)P_{n-1}(\mu)$$

or

$$(n + 1)P_{n+1}(\mu) - (2n + 1)\mu P_n(\mu) + nP_{n-1}(\mu) = 0. \quad (31)$$

Again, let (33) be differentiated with regard to μ; this gives

$$z(1 - 2\mu z + z^2)^{-\frac{3}{2}} = \sum_{1}^{\infty} z^n P'_n(\mu),$$

from which, by comparison with (34), it results that

$$(\mu - z)\sum_{1}^{\infty} z^n P'_n(\mu) = z\sum_{1}^{\infty} n z^{n-1} P_n(\mu).$$

Hence, by equating coefficients, we have

$$\mu P'_n(\mu) - P'_{n-1}(\mu) = nP_n(\mu). \quad . \quad . \quad (35)$$

By differentiating (31), and eliminating $\mu P'_n(\mu)$ from the resulting equation by means of (35), it can be deduced that

$$P'_{n+1}(\mu) - P'_{n-1}(\mu) = (2n + 1)P_n(\mu). \quad . \quad (29)$$

Next, subtract (35) from (29); then

$$P'_{n+1}(\mu) - \mu P'_n(\mu) = (n + 1)P_n(\mu). \quad . \quad (36)$$

Finally, by writing $n - 1$ for n in this equation, and eliminating $P'_{n-1}(\mu)$ between the equation so obtained and (35), it can be shown that

$$(1 - \mu^2)P'_n(\mu) = nP_{n-1}(\mu) - n\mu P_n(\mu) \quad . \quad . \quad . \quad (37)$$

$$= (n + 1)\mu P_n(\mu) - (n + 1)P_{n+1}(\mu), \quad (37')$$

from (31). Formulæ (29), (31), (35), (36), and (37) are the *Recurrence Formulæ.*

Example Deduce (26) and (28) from (29) and (35)

The expansion

$$\frac{1 - z^2}{(1 - 2\mu z + z^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n + 1)z^n P_n(\mu) \quad . \quad (38)$$

can be obtained by multiplying (34) by $2z$ and adding it to (33).

Again, by putting $n = 1, 2, 3, \ldots, n$ in (29), and adding the resulting equations to (30), we derive the summation formula

$$\sum_{m=0}^{n} (2m + 1)P_m(\mu) = P'_{n+1}(\mu) + P'_n(\mu) \quad . \quad (39)$$

But, from (37') and (37)

$$(1 - \mu^2)P'_n(\mu) = (n + 1)\mu P_n(\mu) - (n + 1)P_{n+1}(\mu),$$

$$(1 - \mu^2)P'_{n+1}(\mu) = (n + 1)P_n(\mu) - (n + 1)\mu P_{n+1}(\mu),$$

and, therefore, on adding and dividing by $1 + \mu$, we have

$$(1 - \mu)\{P'_{n+1}(\mu) + P'_n(\mu)\} = (n + 1)\{P_n(\mu) - P_{n+1}(\mu)\}$$

Hence, from (39),

$$(1 - \mu) \sum_{m=0}^{n} (2m + 1)P_m(\mu) = (n + 1)\{P_n(\mu) - P_{n+1}(\mu)\}. \quad (40)$$

Christoffel's First Summation Formula. From (31) we have

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x),$$

$$(2n + 1)yP_n(y) = (n + 1)P_{n+1}(y) + nP_{n-1}(y).$$

Multiply the first of these equations by $P_n(y)$ and the second by $P_n(x)$ and subtract; thus

$$(2n + 1)(x - y)P_n(x)P_n(y) = (n + 1)\{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)\}$$

$$- n\{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)\}.$$

Now in this equation write $n = 1, 2, \ldots, n$, successively, and add the resulting equations; this gives

$$(x-y)\Lambda_n = (n+1)\{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)\}, \qquad (41)$$

where

$$\Lambda_n = P_0(x)P_0(y) + 3P_1(x)P_1(y) + \ldots + (2n+1)P_n(x)P_n(y). \qquad (42)$$

Formula (41) is Christoffel's Summation Formula.

If in (41) and (42) we put $y = x + \epsilon$, (41) becomes

$$-\epsilon\Lambda_n = (n+1)\epsilon\{P_{n+1}(x)P'_n(x) - P_n(x)P'_{n+1}(x)\} + \epsilon^2(\quad).$$

Hence, on dividing by ϵ, and making ϵ tend to zero, we deduce the formula,

$$P_0^2(x) + 3P_1^2(x) + \ldots + (2n+1)P_n^2(x)$$
$$= (n+1)\{P_n(x)P'_{n+1}(x) - P_{n+1}(x)P'_n(x)\}. \qquad (43)$$

Example. Show that the expression on the left of (43) is equal to

$$(n+1)^2P_n^2(x) + (1-x^2)\{P'_n(x)\}^2.$$

§ 9. **Laplace's First Integral.** If ζ is real and numerically < 1, then

$$\int_0^\pi \frac{d\phi}{1+\zeta\cos\phi} = \frac{\pi}{\sqrt{(1-\zeta^2)}} \qquad (44)$$

This equation remains valid so long as ζ is not real and numerically greater than 1; *i.e.*, for the entire complex ζ-plane bounded by cross-cuts along the real axis from 1 to $+\infty$ and from -1 to $-\infty$.

In (44) put $\zeta = \mp\alpha\sqrt{(z^2-1)}\,(1-\alpha z)$ and the equation becomes

$$\int_0^\pi \frac{d\phi}{1-\alpha z \mp \alpha\sqrt{(z^2-1)}\cos\phi} = \frac{\pi}{\sqrt{(1-2\alpha z+\alpha^2)}} \qquad (45)$$

Now choose α so small that $|\alpha\{z \pm \sqrt{(z^2-1)}\cos\phi\}| < 1$, $(0 \leqq \phi \leqq \pi)$, expand both sides of (45) in powers of α, and equate the coefficients of α^n; thus

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \{z \pm \sqrt{(z^2-1)}\cos\phi\}^n d\phi. \qquad (46)$$

This is *Laplace's First Integral.*

If the expression $\{z \pm \sqrt{(z^2 - 1)} \cos \phi\}^n$ be expanded by the binomial theorem and integrated, the terms which contain odd powers of $\sqrt{(z^2 - 1)}$ involve also odd powers of $\cos \phi$, so that their integrals vanish; the right-hand side of (46) is therefore, like the left-hand side, a polynomial in z, and consequently equation (46) is an identity which is valid for all values of z, real or complex.

§ 10. **Laplace's Second Integral.** Again, in (44) put $\zeta = \pm \alpha \sqrt{(z^2 - 1)}/(\alpha z - 1)$; thus

$$\int_0^{\pi} \frac{d\phi}{\alpha z \pm \alpha\sqrt{(z^2 - 1)} \cos \phi - 1} = \frac{\pm \pi}{\sqrt{(\alpha^2 - 2\alpha z + 1)}}. \quad (47)$$

Now choose α so large that $| \alpha\{z \pm \sqrt{(z^2 - 1)} \cos \phi\} | > 1$, $(0 \leqq \phi \leqq \pi)$; then, expanding both sides in descending powers of α and equating coefficients, we have

$$P_n(z) = \pm \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z \pm \sqrt{(z^2 - 1)} \cos \phi\}^{n+1}}.$$

The integrand has a singularity if $z/\sqrt{(z^2 - 1)}$ is real and numerically less than 1. In that case $z^2/(z^2 - 1)$ must be positive and less than 1, and therefore z^2 is negative, and z is purely imaginary. Hence the imaginary axis in the complex z-plane is a *line of singularities* for the integral.

If $z = 1$, $P_n(z) = 1$, so that the positive sign must be taken; if $z = -1$, $P_n(z) = (-1)^n$, and therefore the minus sign must be taken; Hence, for values of z such that $R(z) > 0$,

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z \pm \sqrt{(z^2 - 1)} \cos \phi\}^{n+1}}, \quad . \quad (48)$$

while, for values of z such that $R(z) < 0$,

$$P_n(z) = -\frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z \pm \sqrt{(z^2 - 1)} \cos \phi\}^{n+1}} \quad . \quad (49)$$

§ 11. **Expansion of a Function in a Series of Legendre Polynomials.** If we assume that the function $f(x)$, which satisfies Dirichlet's Conditions (Ch I., § 2) in the interval $(-1, 1)$, can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x), \quad . \quad . \quad . \quad (50)$$

and if we further assume that the series on the right of equation (50) can be integrated term by term over the range $(-1, 1)$, then, exactly as in the case of the polynomial considered in § 7 (*Note*), it can be shown that

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx. \qquad . \qquad . \qquad (51)$$

A proof of the validity of this expansion lies beyond the scope of the present volume; it can be proved, however, by methods similar to those employed in Dirichlet's proof of Fourier's Theorem, that, if $-1 < x < 1$, the sum of the series on the right of (50), with the values of the coefficients given by (51), is $\frac{1}{2}\{f(x-0) + f(x+0)\}$.

Example. 1. Show that

$$\int_0^1 P_n(x)dx = \begin{cases} 1, \text{ if } n = 0, \\ 0, \text{ if } n = 2, 4, 6, \ldots, \\ (-1)^{\frac{n-1}{2}} \dfrac{(n-1)!}{2^n\left(\dfrac{n+1}{2}\right)!\left(\dfrac{n-1}{2}\right)!}, \text{ if } n = 1, 3, 5, \ldots \end{cases}$$

Deduce that, if $f(x) = 0$ for $-1 \leq x < 0$, and $f(x) = 1$ for $0 < x \leq 1$,

$$f(x) = \frac{1}{2} + \frac{3}{2^2}P_1(x) - \frac{7 \cdot 2!}{2^4 \cdot 2!\,1!}P_3(x) + \frac{11 \cdot 4!}{2^6 \cdot 3!\,2!}P_5(x) - \ldots$$

Example 2. By considering the integral $\int_0^1 (1 - 2xh + h^2)^{-\frac{1}{2}}xdx$, show that, if $n > 1$,

$$\int_0^1 xP_n(x)dx = \begin{cases} 0, \text{ if } n \text{ is odd}, \\ (-1)^{\frac{n+2}{2}} \dfrac{(n-2)!}{2^n\left(\dfrac{n}{2}+1\right)!\left(\dfrac{n}{2}-1\right)!}, \text{ if } n \text{ is even}. \end{cases}$$

Deduce that, if $f(x) = 0$ for $-1 \leq x \leq 0$, and $f(x) = x$ for $0 \leq x \leq 1$,

$$f(x) = \frac{1}{2^2} + \frac{1}{2}P_1(x) + \frac{5}{2^4}P_2(x) - \frac{2!\,9}{2^5 \cdot 3!\,1!}P_4(x) + \frac{4!\,13}{2^7 \cdot 4!\,2!}P_6(x) - \ldots$$

Examples

1. Show that, if n is even,

$$P_n(0) = (-1)^{\frac{1}{2}n}\frac{1 \cdot 3 \ldots (n-1)}{2 \cdot 4 \ldots n}, \quad P_n'(0) = 0,$$

while, if n is odd

$$P_n(0) = 0, \ \lim_{\mu \to 0}\{P_n(\mu)/\mu\} = P_n'(0) = (-1)^{\frac{n-1}{2}}\frac{1 \cdot 3 \ldots n}{2 \cdot 4 \ldots (n-1)}.$$

2. Show that

$$\text{(i)}\ P_{2n}(\mu)=(-1)^n\frac{1\cdot3\ \ldots\ (2n-1)}{2\cdot4\ \ldots\ (2n)}F(-n,\ n+\tfrac12,\ \tfrac12,\ \mu^2),$$

$$\text{(ii)}\ P_{2n+1}(\mu)=(-1)^n\frac{3\cdot5\ \ldots\ (2n+1)}{2\cdot4\ \ldots\ (2n)}\mu F(-n,\ n+\tfrac32,\ \tfrac32,\ \mu^2).$$

3. Show that

$$P_n(-\tfrac12) = P_0(-\tfrac12)P_{2n}(\tfrac12) + P_1(-\tfrac12)P_{2n-1}(\tfrac12) + \ldots + P_{2n}(-\tfrac12)P_0(\tfrac12).$$

[Expand both sides of the equation

$$(1 + z^2 + z^4)^{-\frac12} = (1 + z + z^2)^{-\frac12}(1 - z + z^2)^{-\frac12}$$

and equate the coefficients of z^{2n}.]

4. If $\nu = \mu \pm \sqrt{(\mu^2 - 1)}$, show that

$$P_n(\mu) = \frac{(2n)!}{2^{2n}(n!)^2}\nu^{-n}F(\tfrac12,\ -n,\ \tfrac12-n,\ \nu^2).$$

[Expand both sides of the equation

$$(1 - 2\mu z + z^2)^{-\frac12} = (1 - \nu z)^{-\frac12}(1 - \nu^{-1}z)^{-\frac12}$$

in powers of z, and equate the coefficients of z^n.]

5. Prove that

$$\text{(i)}\ (2n+1)(1-\mu^2)P'_n = n(n+1)\{P_{n-1} - P_{n+1}\},$$

$$\text{(ii)}\ \int_{-1}^{1}(\mu^2-1)P_{n+1}P'_n d\mu = \frac{2n(n+1)}{(2n+1)(2n+3)},$$

$$\text{(iii)}\ (m+n+1)\int_0^1\mu^m P_n d\mu = m\int_0^1\mu^{m-1}P_{n-1}d\mu$$
$$= (m-n+2)\int_0^1\mu^m P_{n-2}\,d\mu.$$

6. Show that

$$P_n(\cos\theta) = \cos^n\theta\ F(-\tfrac12 n,\ \tfrac12-\tfrac12 n,\ 1,\ -\tan^2\theta)$$

[In (46) put $z = \cos\theta$, expand in powers of $\cos\phi$, and integrate term by term.]

7. Show that

$$P_n(\cos\theta) = (\cos\tfrac12\theta)^{2n}\ F(-n,\ -n,\ 1,\ -\tan^2\tfrac12\theta).$$

[In 46 put $z = \cos\theta$ and $(\cos\theta + i\sin\theta\cos\phi)^n = (\cos\tfrac12\theta + i\sin\tfrac12\theta\, e^{i\phi})^n(\cos\tfrac12\theta + i\sin\tfrac12\theta\, e^{-i\phi})^n$, expand, and integrate. Since the integrand is symmetrical in $e^{i\phi}$ and $e^{-i\phi}$, when multiplied out it will give terms of the form $\cos m\phi$, where m is an integer, and terms independent of ϕ. Only the latter give non-zero values when integrated.]

8. Prove that, if $(u,\ \phi,\ z)$ and $(r,\ \theta,\ \phi)$ are the cylindrical and polar coordinates of the same point, and if $\mu = \cos\theta$,

$$P_n(\mu) = (-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial z^n}\left(\frac1r\right).$$

[Let $f(u, z) \equiv r^{-1} = (u^2 + z^2)^{-\frac{1}{2}}$, so that $f(u, z - k) = \{u^2 + (z - k)^2\}^{-\frac{1}{2}}$. Keep u constant, and expand by Taylor's Theorem ; then

$$f(u, z - k) = f(u, z) - \frac{k}{1\,!}\frac{\partial}{\partial z}f(u, z) + \ldots + (-1)^n\frac{k^n}{n\,!}\frac{\partial^n}{\partial z^n}f(u, z) + \ldots ;$$

and

$$f(u, z - k) = (r^2 - 2kz + k^2)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - 2\mu\frac{k}{r} + \frac{k^2}{r^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{0}^{\infty}\frac{k^n}{r^n}\mathrm{P}_n(\mu).$$

Equate the coefficients of k^n.]

9. Show that $\mathrm{P}'_{2n+1}(\mu)$ can be expressed in the two following forms :—

(i) $(4n + 1)\mathrm{P}_{2n} + (4n - 3)\mathrm{P}_{2n-2} + \ldots + 5\mathrm{P}_2 + 1$,

(ii) $(2n + 1)\mathrm{P}_{2n} + 2n\mu\mathrm{P}_{2n-1} + (2n - 1)\mu^2\mathrm{P}_{2n-2} + \ldots + 2\mu^{2n-1}\mathrm{P}_1 + \mu^{2n}$.

10. If m and n are distinct positive integers, show that

$$\int_{-1}^{1}(1 - \mu^2)\mathrm{P}'_m(\mu)\mathrm{P}'_n(\mu)d\mu = 0.$$

11. Prove that

(i) $$\int_{-1}^{1}(1 - \mu^2)\{\mathrm{P}'_n(\mu)\}^2 d\mu = \frac{2n(n + 1)}{2n + 1};$$

(ii) $$\int_{-1}^{1}\mu^2\mathrm{P}_{n+1}(\mu)\mathrm{P}_{n-1}(\mu)d\mu = \frac{2n(n + 1)}{(2n - 1)(2n + 1)(2n + 3)};$$

(iii) $$\int_{-1}^{1}\mu^2\{\mathrm{P}_n(\mu)\}^2 d\mu = \frac{1}{8(2n - 1)} + \frac{3}{4(2n + 1)} + \frac{1}{8(2n + 3)}.$$

12. Show that

$$\int_{-1}^{1}\frac{\mathrm{P}_{2n}(x)dx}{(1 + kx^2)^{n+\frac{3}{2}}} = \frac{2}{2n + 1}\frac{(-k)^n}{(1 + k)^{n+\frac{1}{2}}}.$$

[Assume that $|\,k\,| < 1$, expand in powers of k, and use (19).]

13. If m and n are positive integers, show that the integral

$$\int_0^{\pi}\mathrm{P}_n(\cos\theta)\cos m\theta\sin\theta\, d\theta$$

has the value zero if $m < n$ or if $m - n$ is odd, while, if $m \geqq n$ and $m - n$ is even, it has the values

$$-2\frac{\{m - (n - 2)\}\{m - (n - 4)\}\ldots(m - 2)m^2(m + 2)\ldots\{m + (n - 2)\}}{\{m - (n + 1)\}\{m - (n - 1)\}\ldots(m - 1)(m + 1)\ldots\{m + (n + 1)\}},\ n \text{ even},$$

$$-2\frac{\{m - (n - 2)\}\{m - (n - 4)\}\ldots(m - 1)(m + 1)\ldots\{m + (n - 2)\}}{\{m - (n + 1)\}\{m - (n - 1)\}\ldots(m - 2)(m + 2)\ldots\{m + (n + 1)\}},\ n \text{ odd},$$

or, for n either odd or even,

$$-2m\frac{\{m - (n - 2)\}\{m - (n - 4)\}\ldots\{m + (n - 2)\}}{\{m - (n + 1)\}\{m - (n - 1)\}\ldots\{m + (n + 1)\}},$$

the factor m being omitted when n is zero.

[From (9), $P_n(\cos\theta) = C_n \cos n\theta + C_{n-2}\cos(n-2)\theta + \dots$, so that, if I denotes the integral,

$$I = \tfrac{1}{2}C_n\left\{\frac{1}{m+n+1} - \frac{1}{m-n-1} - \frac{1}{m+n-1} + \frac{1}{m-n+1}\right\} + \dots$$

Hence I = (Polynomial in m) ÷ (Polynomial in m), the denominator being of one of the forms given in the first two formulæ, and the numerator being an even polynomial of degree 2 less than the denominator. This holds for all even integral values of $m - n$, and the factors of the numerator can be deduced from the fact that $I = 0$ when $m = \pm(n-2), \pm(n-4) \dots$ The constant factor can be determined from the fact that

$$\lim_{m \to \infty} m^2 I = -2(C_n + C_{n-2} + \dots),$$

since, when $\theta = 0$, $P_n(1) = 1 = C_n + C_{n-2} + \dots$.]

14. Show that

$$\cos m\theta = B_m P_m(\cos\theta) + B_{m-2}P_{m-2}(\cos\theta) + \dots,$$

where

$$B_n = -(2n+1)m\frac{\{m-(n-2)\}\{m-(n-4)\}\dots\{m+(n-2)\}}{\{m-(n+1)\}\{m-(n-1)\}\dots\{m+(n+1)\}}.$$

15. Verify that

$$\cos\theta = P_1,\ \cos 2\theta = \tfrac{4}{3}P_2 - \tfrac{1}{3}P_0,\ \cos 3\theta = \tfrac{8}{5}P_3 - \tfrac{3}{5}P_1,$$

$$\cos 4\theta = \tfrac{64}{35}P_4 - \tfrac{16}{21}P_2 - \tfrac{1}{15}P_0,\ \cos 5\theta = \tfrac{128}{63}P_5 - \tfrac{8}{9}P_3 - \tfrac{1}{7}P_1,$$

$$\cos 6\theta = \tfrac{512}{231}P_6 - \tfrac{384}{385}P_4 - \tfrac{4}{21}P_2 - \tfrac{1}{35}P_0,$$

$$\cos 7\theta = \tfrac{1024}{429}P_7 - \tfrac{128}{117}P_5 - \tfrac{112}{495}P_3 - \tfrac{1}{15}P_1,$$

$$\cos 8\theta = \tfrac{16384}{6435}P_8 - \tfrac{4096}{3465}P_6 - \tfrac{256}{1001}P_4 - \tfrac{64}{693}P_2 - \tfrac{1}{63}P_0,$$

$$\cos 9\theta = \tfrac{32768}{12155}P_9 - \tfrac{3072}{2431}P_7 - \tfrac{128}{455}P_5 - \tfrac{16}{143}P_3 - \tfrac{3}{77}P_1,$$

$$\cos 10\theta = \tfrac{131072}{46189}P_{10} - \tfrac{32768}{24453}P_8 - \tfrac{512}{1683}P_6 - \tfrac{128}{1001}P_4 - \tfrac{500}{9009}P_2 - \tfrac{1}{99}P_0.$$

16. If m and n are positive integers, show that the integral

$$\int_0^\pi P_n(\cos\theta)\sin m\theta\, d\theta$$

has the value zero if $n \geqq m$ or if $m + n$ is even, while, if $m > n$ and $m + n$ is odd, it is equal to

$$2\frac{(m-n+1)(m-n+3)\dots(m+n-1)}{(m-n)(m-n+2)\dots(m+n)}.$$

17. From the previous example deduce that, for $0 < \theta < \pi$,

$$\frac{\pi}{4}P_n(\cos\theta) = \frac{2\cdot 4\dots(2n)}{1\cdot 3\cdot 5\dots(2n+1)}\sin(n+1)\theta$$

$$+\frac{4\cdot 6\dots(2n+2)}{3\cdot 5\dots(2n+3)}\sin(n+3)\theta + \frac{6\cdot 8\dots(2n+4)}{5\cdot 7\dots(2n+5)}\sin(n+5)\theta + \dots$$

18. Show that, if m and s are positive integers,

$$\int_0^\pi P_{m+2s}(\cos\theta)\cos(m\theta)\, d\theta = \frac{\Gamma(s+\frac{1}{2})\Gamma(m+s+\frac{1}{2})}{\Gamma(s+1)\Gamma(m+s+1)}.$$

[Use formula (9).]

19. Employ the transformation

$$(\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi)(\mu \mp \sqrt{(\mu^2 - 1)} \cos \psi) = 1$$

to prove that Laplace's First and Second Integrals are equal.

20. If $f(\mu)$ is a polynomial of degree n such that $\int_{-1}^{1} \mu^m f(\mu) d\mu = 0$ for $m = 0, 1, 2, \ldots, n - 1$, show that $f(\mu)$ is a constant multiple of $P_n(\mu)$.

[Let $f_1(\mu) = \int_{-1}^{\mu} f(\mu) d\mu$, $f_2(\mu) = \int_{-1}^{\mu} f_1(\mu)$, . . . ; then, by partial integration,

$$\int_{-1}^{\mu} \mu^m f(\mu) d\mu = f_1(\mu)\mu^m - m f_2(\mu)\mu^{m-1} + \ldots + (-1)^m . m! f_{m+1}(\mu).$$

Now put $m = 0, 1, \ldots, n - 1$, and deduce that $f_1(\mu), f_2(\mu), \ldots, f_n(\mu)$ all vanish when $\mu = 1$. Thus $f_n(\mu)$ and its first $n - 1$ derivatives vanish when $\mu = 1$. But this is also true when $\mu = -1$, and $f_n(\mu)$ is of degree $2n$ Therefore

$$f_n(\mu) = C(\mu - 1)^n(\mu + 1)^n = C(\mu^2 - 1)^n, \text{ and } f(\mu) = \frac{d^n}{d\mu^n} f_n(\mu).]$$

21. If m is a positive integer, and

$$(1 - 2\mu z + z^2)^{-m-\frac{1}{2}} = \sum_{n=0}^{\infty} P_{m,n}(\mu) z^n$$

show that

$$P_{m,n}(\mu) = \frac{2^m . m!}{(2m)!} \frac{d^m}{d\mu^m} P_{m+n}(\mu).$$

Deduce that $P_{m,n}(\mu)$ satisfies the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2(m + 1)x\frac{dy}{dx} + n(2m + n + 1)y = 0.$$

[Differentiate the equation $(1 - 2\mu z + z^2)^{-\frac{1}{2}} = \Sigma P_n(\mu) z^n$ m times with regard to μ. For the second part put $m + n$ for n in Legendre's Equation and differentiate m times.]

22. Prove that formulæ (19) and (20) hold for any values of m for which the integrals are convergent.

[For example, if r is even, let $P_r(\mu) = C_0\mu^r + C_2\mu^{r-2} + \ldots + C_r$, so that $P_r(1) = 1 = C_0 + C_2 + \ldots + C_r$. Then

$$I = C_0 \frac{1}{m + r + 1} + C_2 \frac{1}{m + r - 1} + \ldots + C_r \frac{1}{m + 1}$$

$$= K \frac{m(m - 2) \ldots (m - r + 2)}{(m + r + 1)(m + r - 1) \ldots (m + 1)},$$

since I vanishes when $m = 0, 2, \ldots r - 2$. But

$$\lim_{m \to \infty} mI = C_0 + C_2 + \ldots + C_r = 1,$$

and therefore K = 1.]

CHAPTER VI

THE LEGENDRE FUNCTIONS

§ 1. **Legendre Functions of the Second Kind.** Legendre's Equation (IV., 12)

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0, \qquad . \quad (1)$$

where n is any number (not necessarily a positive integer) has, according to the theory of linear differential equations, two linearly independent integrals. It is found convenient to obtain one of these in the form of a series of descending powers of x.

Let $y = x^\rho(c_0 + c_1x^{-1} + c_2x^{-2} + \dots)$

and substitute this expression for y in the left-hand side of (1): then

$$\begin{aligned}
&(1 - x^2)y'' - 2xy' + n(n + 1)y \\
&\quad = - x^\rho c_0\{\rho(\rho - 1) + 2\rho - n(n + 1)\} \\
&\qquad - x^{\rho - 1}c_1\{(\rho - 1)(\rho - 2) + 2(\rho - 1) - n(n + 1)\} \\
&+ x^{\rho - 2}[c_0\rho(\rho - 1) - c_2\{(\rho - 2)(\rho - 3) + 2(\rho - 2) - n(n + 1)\}] \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&+ x^{\rho - \nu}[c_{\nu - 2}(\rho - \nu + 2)(\rho - \nu + 1) \\
&\qquad\qquad - c_\nu\{(\rho - \nu)(\rho - \nu - 1) + 2(\rho - \nu) - n(n + 1)\}] \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&= - x^\rho c_0(\rho - n)(\rho + n + 1) - x^{\rho - 1}c_1(\rho - n - 1)(\rho + n) \\
&\quad + x^{\rho - 2}[c_0\rho(\rho - 1) - c_2(\rho - n - 2)(\rho + n - 1)] \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&+ x^{\rho - \nu}[c_{\nu - 2}(\rho - \nu + 2)(\rho - \nu + 1) - c_\nu(\rho - n - \nu)(\rho + n - \nu + 1)] \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned}$$

If the coefficients of the different powers of x in this expression all vanish, Legendre's Equation will be formally satisfied by the series for y: if, moreover, the series is convergent, it will define a function which is a solution of the differential equation.

The equations obtained by equating the coefficients to zero are

$$\begin{aligned}
&c_0(\rho - n)(\rho + n + 1) = 0,\\
&c_1(\rho - n - 1)(\rho + n) = 0,\\
&c_2(\rho - n - 2)(\rho + n - 1) = c_0\rho(\rho - 1),\\
&c_3(\rho - n - 3)(\rho + n - 2) = c_1(\rho - 1)(\rho - 2), \qquad (2)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&c_\nu(\rho - n - \nu)(\rho + n - \nu + 1) = c_{\nu-2}(\rho - \nu + 2)(\rho - \nu + 1),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Since c_0 is the coefficient of the first term in the expansion it must have a non-zero value; the first of these equations can therefore be written

$$(\rho - n)(\rho + n + 1) = 0. \quad . \quad . \quad . \quad (3)$$

This is known as the *indicial equation*, as the solutions $\rho_1 = n$ and $\rho_2 = -n - 1$ give the two possible values of the index of the first term in the series for y.

The coefficient of c_ν in the $(\nu + 1)$th equation of (2) is of the form

$$(\rho - \rho_1 - \nu)(\rho - \rho_2 - \nu).$$

If $\rho = \rho_1$ or $\rho = \rho_2$ this coefficient will not vanish unless ρ_1 and ρ_2 differ by an integer; *i.e.*, unless $2n + 1$ is an integer, or, what is the same thing, unless n is an integer or half an odd integer. It will be assumed, to begin with, that $2n + 1$ *is not an integer.*

Accordingly, from the second equation of (2) it follows that $c_1 = 0$: hence, from the fourth equation, $c_3 = 0$, and similarly $c_\nu = 0$ whenever ν is odd. Again,

$$\begin{aligned}
c_2 &= c_0\frac{\rho(\rho - 1)}{(\rho - n - 2)(\rho + n - 1)},\\
c_4 &= c_2\frac{(\rho - 2)(\rho - 3)}{(\rho - n - 4)(\rho + n - 3)}\\
&= c_0\frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{(\rho - n - 2)(\rho - n - 4)(\rho + n - 1)(\rho + n - 3)},
\end{aligned}$$

and so on. Thus the two solutions which correspond respectively to $\rho = n$ and $\rho = -n - 1$ are

$$y_1 = c_0 x^n\left\{1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}x^{-4} - \ldots\right\}$$

$$= c_0 x^n F\left(-\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2} - n, x^{-2}\right). \quad . \quad . \quad . \quad (4)$$

and

$$y_2 = c_0 x^{-n-1}\left\{1 + \frac{(n+1)(n+2)}{2(2n+3)}x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4(2n+3)(2n+5)}x^{-4} + \ldots\right\}$$

$$= c_0 x^{-n-1}F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{3}{2} + n, x^{-2}\right) \quad . \quad . \quad (5)$$

Let the function $Q_n(x)$ be defined by means of the equation

$$Q_n(x) = B(\tfrac{1}{2}, n+1)\frac{1}{(2x)^{n+1}}F\left(\frac{n+1}{2}, \frac{n+2}{2}, n+\frac{3}{2}, \frac{1}{x^2}\right); \quad (6)$$

then, when $2n + 1$ is not an integer, $Q_n(x)$ is a multiple of y_2, and is therefore a solution of Legendre's Equation.

If n is an *integer* or *zero*, all the coefficients in y_1 and y_2 are finite,* and therefore both solutions still satisfy the differential equation. In particular, if n is zero or a positive integer, y_1 is (cf. V., 6′) a multiple of $P_n(x)$. If, however, n tends to a negative integral value, $\Gamma(n + 1)$ and therefore $B(\frac{1}{2}, n + 1)$ tend to ∞, so that the function $Q_n(x)$ ceases to have any value.

If n tends to *half an odd positive integer*, some of the coefficients in the expansion y_1 tend to ∞, so that this solution no longer exists. On the other hand, the coefficients in the expansion y_2 remain finite, so that $Q_n(x)$ is a solution in this case also.

Finally, if n is *half an odd negative integer*, the coefficients in y_1 remain finite. But, in the expansion (cf. IV., 45, 39)

$$\frac{\sqrt{\pi}}{(2x)^{n+1}}\sum_{r=0}^{\infty}\frac{\Gamma(n+2r+1)}{r!\;\Gamma(n+\frac{3}{2}+r)}\frac{1}{(2x)^{2r}}. \quad (6a)$$

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* There may, of course, be a zero factor in the denominator of the expression obtained from (2) for c_ν (ν odd) in terms of c_0, but this will be cancelled by a similar factor in the numerator.

for $Q_n(x)$ the factors $1/\Gamma(n + \frac{3}{2} + r)$ are zero if $n + \frac{3}{2} + r$ is zero or a negative integer; hence the first $-(n + \frac{1}{2})$ terms in the expansion vanish, the first non-zero term being that for which $r = -n - \frac{1}{2}$. Thus

$$Q_n(x) = \frac{\sqrt{\pi}\Gamma(-n)}{(2x)^{-n}\Gamma(-n + \frac{1}{2})}F\left(-\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2} - n, x^{-2}\right),$$

which is a constant multiple of y_1.

It has, therefore, been shown that $Q_n(x)$ is a solution of Legendre's Equation, unless when n is a negative integer. The series for $Q_n(x)$ converges when $|x| > 1$, and the function $Q_n(x)$ is known as the *Legendre Function of degree* n *of the Second Kind.*

Example. Verify that

$$\text{(i)}\ Q_0(x) = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right),$$

$$\text{(ii)}\ Q_1(x) = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) - 1.$$

[Expand the right-hand sides in descending powers of x, and compare with formula (6).]

§ 2. **Legendre Functions of the First Kind.** The substitution $x = 1 - 2\xi$ transforms Legendre's Equation (1) into the equation

$$\xi(1 - \xi)\frac{d^2y}{d\xi^2} + (1 - 2\xi)\frac{dy}{d\xi} + n(n + 1)y = 0,$$

which is Gauss's Equation (IV., 19) with $\alpha = n + 1$, $\beta = -n$, $\gamma = 1$. Now it has been shown that, for all values of n, $F(n + 1, -n, 1, \xi)$ is a solution of this equation; thus Legendre's Equation is always satisfied by the function $F\left(n + 1, -n, 1, \frac{1-x}{2}\right)$.

But, when n is a positive integer, this is the function $P_n(x)$ (cf. V., 7). We therefore, for all values of n, define $P_n(x)$, the *Legendre Function of degree* n *of the First Kind*, by means of the equation

$$P_n(x) = F\left(-n, n + 1, 1, \frac{1-x}{2}\right). \quad . \quad . \quad (7)$$

This series is convergent for $|x - 1| < 2$. Since

$$F\left(n + 1, -n, 1, \frac{1-x}{2}\right) \equiv F\left(-n, n + 1, 1, \frac{1-x}{2}\right),$$

it follows that

$$P_n(x) = P_{-n-1}(x) \quad . \quad . \quad . \quad (8)$$

$P_n(x)$ is the more important of the Legendre Functions when $|x| < 1$ and $Q_n(x)$ when $|x| > 1$.

§ 3. **The Recurrence Formulae for $P_n(x)$** From (7) we have, for all values of n,

$$(n + 1)P_{n+1}(x) - (2n + 1)P_n(x) + nP_{n-1}(x)$$

$$= (n + 1) \sum_{r=0}^{\infty} \frac{(n - r + 2)(n - r + 3) \ldots (n + r + 1)}{(r!)^2}\left(\frac{x-1}{2}\right)^r$$

$$- (2n + 1) \sum_{r=0}^{\infty} \frac{n - r + 1)(n - r + 2) \ldots (n + r)}{(r!)^2}\left(\frac{x-1}{2}\right)^r$$

$$+ n \sum_{r=0}^{\infty} \frac{(n - r)(n - r + 1) \ldots (n + r - 1)}{(r!)^2}\left(\frac{x-1}{2}\right)^r$$

$$= \sum_{r=1}^{\infty} [(n + 1)(n + r)(n + r + 1) - (2n + 1)(n - r + 1)(n + r) + n(n - r)(n - r + 1)]$$

$$\times \frac{(n - r + 2)(n - r + 3) \ldots (n + r - 1)}{(r!)^2}\left(\frac{x-1}{2}\right)^r.$$

The expression in the square bracket is equal to $2(2n + 1)r^2$; hence

$$(n + 1)P_{n+1}(x) - (2n + 1)P_n(x) + nP_{n-1}(x) = (2n + 1)(x - 1)P_n(x),$$

so that

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0. \quad (9)$$

Again, $\qquad P'_{n+1}(x) - P'_{n-1}(x)$

$$= \sum_{r=1}^{\infty} \frac{(n - r + 2)(n - r + 3) \ldots (n + r + 1)}{2r\{(r - 1)!\}^2}\left(\frac{x-1}{2}\right)^{r-1}$$

$$-\sum_{r=1}^{\infty}\frac{(n-r)(n-r+1)\ldots(n+r-1)}{2r\{(r-1)!\}^2}\left(\frac{x-1}{2}\right)^{r-1}$$

$$=\sum_{r=1}^{\infty}[(n+r)(n+r+1)-(n-r)(n-r+1)]$$

$$\times\frac{(n-r+2)(n-r+3)\ldots(n+r-1)}{2r\{(r-1)!\}^2}\left(\frac{x-1}{2}\right)^{r-1}.$$

The expression in the square bracket is equal to $(2n+1)(2r)$; hence

$$P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x). \quad . \quad (10)$$

Again, differentiate (9) and substitute for $P'_{n-1}(x)$ from (10); thus

$$P'_{n+1}(x)-xP'_n(x)=(n+1)P_n(x). \quad . \quad (11)$$

Next, subtract (11) from (10); then

$$xP'_n(x)-P'_{n-1}(x)=nP_n(x). \quad . \quad . \quad (12)$$

Finally, multiply (12) by x, write $n-1$ for n in (11), and subtract; thus

$$(x^2-1)P'_n(x)=nxP_n(x)-nP_{n-1}(x) \quad . \quad (13)$$

§ 4. **The Recurrence Formula for** $Q_n(x)$. From (6) $nQ_{n-1}(x)-(2n+1)xQ_n(x)$

$$=\frac{\sqrt{\pi}\,.\,\Gamma(n+1)}{2^n\Gamma(n+\frac{1}{2})}\frac{1}{x^n}\sum_{r=1}^{\infty}\left\{\frac{n(n+1)\ldots(n+2r-1)}{r!(2n+1)(2n+3)\ldots(2n+2r-1)}-\frac{(n+1)(n+2)\ldots(n+2r)}{r!(2n+3)(2n+5)\ldots(2n+2r+1)}\right\}\frac{1}{2^rx^{2r}}$$

$$=\frac{\sqrt{\pi}\,.\,\Gamma(n+2)}{2^{n+2}\Gamma(n+\frac{5}{2})}\frac{1}{x^{n+2}}\sum_{r=1}^{\infty}[n(2n+2r+1)-(n+2r)(2n+1)]$$

$$\times\frac{(n+2)(n+3)\ldots(n+2r-1)}{r!(2n+5)(2n+7)\ldots(2n+2r+1)}\frac{1}{2^{r-1}x^{2r-2}}.$$

The expression in the square bracket is equal to $-2r(n+1)$; hence, if $n \neq 0$,

$$(n+1)Q_{n+1}(x)-(2n+1)xQ_n(x)+nQ_{n-1}(x)=0, \quad (14)$$

while, if $n=0$,

$$Q_1(x)-xQ_0(x)+1=0. \quad . \quad . \quad (15)$$

Again

$$Q'_{n-1}(x) + (2n+1)Q_n(x)$$

$$= -\frac{\sqrt{\pi}\,.\,\Gamma(n+1)}{2^n\Gamma(n+\frac{1}{2})}\frac{1}{x^{n+1}}$$

$$\times \sum_{r=1}^{\infty}\left\{\begin{aligned}&\frac{(n+1)(n+2)\ldots(n+2r-1)}{r!\,(2n+1)(2n+3)\ldots(2n+2r-1)}\frac{(n+2r)}{2^r x^{2r}}\\ &-\frac{(n+1)(n+2)\ldots(n+2r)}{r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}\frac{1}{2^r x^{2r}}\end{aligned}\right\}$$

$$= -\frac{\sqrt{\pi}\,.\,\Gamma(n+2)}{2^{n+2}\Gamma(n+\frac{5}{2})}\frac{1}{x^{n+2}}$$

$$\times \sum_{r=1}^{\infty}\frac{(n+2)(n+3)\ldots(n+2r-1)}{(r-1)!\,(2n+5)\ldots(2n+2r+1)}\frac{n+2r}{2^{r-1}x^{2r-1}}$$

$$= Q'_{n+1}(x).$$

Thus

$$Q'_{n+1}(x) - Q'_{n-1}(x) = (2n+1)Q_n(x). \quad . \quad (16)$$

As in § 3 it can be deduced that

$$Q'_{n+1}(x) - xQ'_n(x) = (n+1)Q_n(x), \quad . \quad (17)$$

$$xQ'_n(x) - Q'_{n-1}(x) = nQ_n(x), \quad . \quad . \quad (18)$$

$$(x^2-1)Q'_n(x) = nxQ_n(x) - nQ_{n-1}(x). \quad . \quad . \quad (19)$$

Example 1. Show that

(i) $Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1} - \frac{3}{2}x,$

(ii) $Q_3(x) = \frac{1}{2}P_3(x)\log\frac{x+1}{x-1} - \frac{5}{2}x^2 + \frac{2}{3}.$

[Use (14) and § 1, *example.*]

Example 2. Show that, if $n \neq 0$,

(i) $(n+1)\{P_{n+1}Q_n - Q_{n+1}P_n\} = n\{P_nQ_{n-1} - Q_nP_{n-1}\},$

(ii) $(n+1)\{P_{n+1}Q_{n-1} - Q_{n+1}P_{n-1}\} = (2n+1)x\{P_nQ_{n-1} - Q_nP_{n-1}\}.$

[Use (9) and (14).]

§ 5. **Theorem.** For all values of n,

$$(x^2-1)\{Q_n(x)P'_n(x) - P_n(x)Q'_n(x)\} = C, \quad . \quad (20)$$

where C is independent of x.

Since $P_n(x)$ and $Q_n(x)$ both satisfy Legendre's Equation,

$$\frac{d}{dx}\{(1-x^2)P'_n(x)\} + n(n+1)P_n(x) = 0,$$

$$\frac{d}{dx}\{(1-x^2)Q'_n(x)\} + n(n+1)Q_n(x) = 0.$$

Multiply the first of these equations by $Q_n(x)$, the second by $P_n(x)$, and subtract; thus

$$Q_n(x)\frac{d}{dx}\{(1-x^2)P'_n(x)\} - P_n(x)\frac{d}{dx}\{(1-x^2)Q'_n(x)\} = 0.$$

Therefore

$$\begin{aligned}(1-x^2)\{Q_n(x)P''_n(x) - P_n(x)Q''_n(x)\}&\\ - 2x\{Q_n(x)P'_n(x) - P_n(x)Q'_n(x)\} &= 0,\end{aligned}$$

or $$\frac{d}{dx}[(1-x^2)\{Q_n(x)P'_n(x) - P_n(x)Q'_n(x)\}] = 0,$$

from which the theorem follows.

Corollary. If n is zero or a positive integer, $C = 1$. This can be established by replacing $Q_n(x)$ and $P_n(x)$ by the series (6) and (V., 6) and making x tend to ∞.

Example. If n is a positive integer, show that

$$Q_n(x) = P_n(x)\int_x^\infty \frac{dx}{(x^2-1)\{P_n(x)\}^2}.$$

[From (20) it follows that $\frac{d}{dx}\{Q_n(x)/P_n(x)\} = -(x^2-1)^{-1}\{P_n(x)\}^{-2}$.]

Christoffel's Second Summation Formula. From (9) and (14) we have

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$

and $$(n+1)Q_{n+1}(y) - (2n+1)yQ_n(y) + nQ_{n-1}(y) = 0.$$

Multiply the first equation by $Q_n(y)$, the second by $P_n(x)$, and subtract; thus

$$\begin{aligned}&(n+1)\{Q_n(y)P_{n+1}(x) - P_n(x)Q_{n+1}(y)\} \\ &= n\{Q_{n-}(y)P_n(x) - P_{n-1}(x)Q_n(y)\} + (2n+1)(x-y)P_n(x)Q_n(y).\end{aligned} \tag{21}$$

Similarly from (V., 32) and (15)

$$P_1(x) - xP_0(x) = 0,$$
$$Q_1(y) - yQ_0(y) + 1 = 0,$$

so that $$\{Q_0(y)P_1(x) - P_0(x)Q_1(y)\} = 1 + (x-y)P_0(x)Q_0(y). \tag{22}$$

In (21) put $n = 1, 2, 3, \ldots, n$, and add the resulting equations and (22): thus

$$\left.\begin{aligned}&(y-x)\Lambda_n = 1-(n+1)\{P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\},\\ &\text{where}\\ &\qquad\Lambda_n = P_0(x)Q_0(y)+3P_1(x)Q_1(y)+\ \ldots\\ &\qquad\qquad\qquad +(2n+1)P_n(x)Q_n(y).\end{aligned}\right\}\quad(23)$$

This is Christoffel's Second Summation Formula.

Examples.

1. If n is zero or a positive integer, show that

$$Q_n(x) = \tfrac{1}{2}\int_{-1}^{1}\frac{P_n(\xi)}{x-\xi}d\xi, \text{ (Neumann's Formula).}$$

[Expand $(x-\xi)^{-1}$ in descending powers of x for $|x| > 1$, and use *Theorem III.* of § 6, Chap. V.]

2. Verify that, when n is a positive integer,

$$Q_n(x) = 2^n\,.\,n!\int_x^\infty\int_v^\infty\int_v^\infty\ldots\int_v^\infty\frac{1}{(v^2-1)^{n+1}}(dv)^{n+1}.$$

3. Show that the substitution $\nu = x \pm \sqrt{(x^2-1)}$ transforms Legendre's Equation into

$$\nu^2(1-\nu^2)\frac{d^2y}{d\nu^2} - 2\nu^3\frac{dy}{d\nu} - n(n+1)(1-\nu^2)y = 0,$$

and that the further transformation $\lambda = \nu^2$ changes it into

$$4\lambda^2(1-\lambda)\frac{d^2y}{d\lambda^2} + 2\lambda(1-3\lambda)\frac{dy}{d\lambda} - n(n+1)(1-\lambda)y = 0.$$

Deduce that any solution of Legendre's Equation can be put in the form

$$A\nu^{-n}F(\tfrac{1}{2}, -n, \tfrac{1}{2}-n, \nu^2) + B\nu^{n+1}F(\tfrac{1}{2}, n+1, n+\tfrac{3}{2}, \nu^2).$$

If n is a positive integer, and if $\xi = x + \sqrt{(x^2-1)}$, where amp $\sqrt{(x^2-1)} = 0$ when x is real and > 1, prove that

$$\begin{aligned}Q_n(x) &= 2\frac{2\,.\,4\ \ldots\ (2n)}{3\cdot5\ \ldots\ (2n+1)}\xi^{-n-1}F\left(\tfrac{1}{2}, n+1, n+\tfrac{3}{2}, \frac{1}{\xi^2}\right)\\ &= B(n+1, \tfrac{1}{2})\xi^{-n-1}F\left(\tfrac{1}{2}, n+1, n+\tfrac{3}{2}, \frac{1}{\xi^2}\right).\end{aligned}$$

4. Prove that $Q_n(x) = \displaystyle\int_0^\infty\frac{du}{\{x+\sqrt{(x^2-1)}\cosh u\}^{n+1}}$.

[From (IV., 27) and *example* 3,

$$Q_n(x) = \xi^{-n-1}\int_0^1 t^{-\frac{1}{2}}(1-t)^n(1-t/\xi^2)^{-n-1}\,dt.$$

Here put $t = (v-1)/(v+1)$, so that $v = (1+t)/(1-t)$; then $Q_n(x) = 2^{n+1}\displaystyle\int_1^\infty\left\{\left(\xi+\frac{1}{\xi}\right)+\left(\xi-\frac{1}{\xi}\right)v\right\}^{-n-1}\frac{dv}{\sqrt{(v^2-1)}}$. Put $v = \cosh u$.]

5. If n is a positive integer, prove that

$$Q_n(x) = \frac{(-2)^n\Gamma(n+1)}{\Gamma(2n+1)}\frac{d^n}{dx^n}\left[(x^2-1)^n\int_x^\infty\frac{dx}{(x^2-1)^{n+1}}\right].$$

[Show that $(x^2-1)^n$ is a solution of

$$(1-x^2)\frac{d^2y}{dx^2}+2(n-1)x\frac{dy}{dx}+2ny=0$$

By means of the substitution $y = v(x^2-1)^n$ deduce that the second solution is $(x^2-1)^n\int_x^\infty\frac{dx}{(x^2-1)^{n+1}}$. Then differentiate the equation n times, and get Legendre's Equation.]

6. Show that

$$\int_x^1 P_m(x)P_n(x)dx = \frac{mP_n(x)P_{m-1}(x)-nP_m(x)P_{n-1}(x)-(m-n)xP_m(x)P_n(x)}{(m-n)(m+n+1)},$$

deduce that, if m and n are positive integers, both odd or both even,

$$\int_0^1 P_m(x)P_n(x)dx = 0,$$

while if m is even and n odd

$$\int_0^1 P_m(x)P_n(x)dx = (-1)^{\frac{1}{2}(m+n+1)}\frac{m!\,n!}{2^{m+n-1}(m-n)(m+n+1)\left(\frac{m}{2}!\right)^2\left(\frac{n-1}{2}!\right)^2}$$

[Use the method employed in the second proof (p. 94) of *Theorem I*, § 6, Chap. V., the formula (13), and *ex.* 1, Chap. V.]

7. If n is zero or a positive integer, show that

$$Q_n(x) = \tfrac{1}{2}P_n(x)\log\left(\frac{x+1}{x-1}\right) - W_{n-1}(x),$$

where $W_{n-1}(x)$ is a polynomial of degree $n-1$.

[In *ex.* 1 write $Q_n(x) = \frac{1}{2}\int_{-1}^1\frac{P_n(x)}{x-\xi}d\xi - \frac{1}{2}\int_{-1}^1\frac{P_n(x)-P_n(\xi)}{x-\xi}d\xi$,

and in the second integral divide out by $x-\xi$ before integrating.]

8. With the notation of *ex.* 7, show that

$$W_{n-1}(x) = \frac{2n-1}{1\cdot n}P_{n-1}(x)+\frac{2n-5}{3(n-1)}P_{n-3}(x)+\ldots.$$

[Substitute the expression obtained for $Q_n(x)$ in *ex.* 7 in Legendre's Equation, put $W_{n-1}(x) = a_1P_{n-1}(x)+a_3P_{n-3}(x)+\ldots$ and use formula (V., 26).]

9. **Show that, if $|x - 1| < 2$,**

$$Q_n(x) = AP_n(x) \log \left(\frac{1 - x}{2}\right) + R_n\left(\frac{1 - x}{2}\right),$$

where A is not zero and $R_n\left(\frac{1 - x}{2}\right)$ is a convergent series of powers of $\left(\frac{1 - x}{2}\right)$.

In the differential equation

$$\xi(1 - \xi)\frac{d^2y}{d\xi^2} + (1 - 2\xi)\frac{dy}{d\xi} + n(n + 1)y = 0 \quad . \quad . \quad (A)$$

considered in § 2, put

$$y = \xi^\rho \sum_{r=0}^{\infty} c_r \xi^r ;$$

then
$$\xi(1 - \xi)\frac{d^2y}{d\xi^2} + (1 - 2\xi)\frac{dy}{d\xi} + n(n + 1)y$$

$$= (1 - \xi)\xi^{\rho - 1}\sum_{r=0}^{\infty} c_r(\rho + r)(\rho + r - 1)\xi^r + (1 - 2\xi)\xi^{\rho - 1}\sum_{r=0}^{\infty} c_r(\rho + r)\xi^r$$

$$+ n(n + 1)\xi^\rho \sum_{r=0}^{\infty} c_r \xi^r$$

$$= \xi^{\rho - 1}\left\{c_0\rho^2 + \sum_{r=1}^{\infty} c_r(\rho + r)^2\xi^r\right\} - \xi^\rho \sum_{r=0}^{\infty} c_r(\rho + r - n)(\rho + r + n + 1)\xi^r$$

$$= c_0\rho^2\xi^{\rho - 1},$$

provided that

$$c_{r+1}(\rho + r + 1)^2 = c_r(\rho + r - n)(\rho + r + n + 1),\ r = 0, 1, 2, \ldots$$

Thus

$$y = c_0\xi^\rho\left\{1 + \frac{(\rho - n)(\rho + n + 1)}{(\rho + 1)^2}\xi + \frac{(\rho - n)(\rho - n + 1)(\rho + n + 1)(\rho + n + 2)}{(\rho + 1)^2(\rho + 2)^2}\xi^2 + \ldots\right\}$$

is a solution of the equation

$$\xi(1 - \xi)\frac{d^2y}{d\xi^2} + (1 - 2\xi)\frac{dy}{d\xi} + n(n + 1)y = c_0\rho^2\xi^{\rho - 1}; \quad . \quad (B)$$

and, by differentiating this equation with regard to ρ, we see that $z = \frac{\partial y}{\partial \rho}$ is a solution of the equation

$$\xi(1 - \xi)\frac{d^2z}{d\xi^2} + (1 + 2\xi)\frac{dz}{d\xi} + n(n + 1)z = c_0\rho\xi^{\rho - 1}(2 + \rho \log \xi) \quad . \quad (C)$$

Now

$$z = y\log\xi + c_0\xi^\rho\left\{\begin{aligned}&\frac{(\rho-n)(\rho+n+1)}{(\rho+1)^2}\left(\frac{1}{\rho-n}+\frac{1}{\rho+n+1}-\frac{2}{\rho+1}\right)\xi\\&+\frac{(\rho-n)(\rho-n+1)(\rho+n+1)(\rho+n+2)}{(\rho+1)^2(\rho+2)^2}\\&\qquad\times\left(\begin{aligned}&\frac{1}{\rho-n}+\frac{1}{\rho-n+1}+\frac{1}{\rho+n+1}\\&+\frac{1}{\rho+n+2}-\frac{2}{\rho+1}-\frac{2}{\rho+2}\end{aligned}\right)\xi^2+\ \ldots\end{aligned}\right\}$$

If the value of ρ be taken to be zero, equations (B) and (C) both reduce to equation (A), of which y gives one solution

$$y_1 = F(-n,\ n+1,\ 1,\ \xi) = P_n(x),$$

by (7), and z gives a second solution

$$y_2 = y_1\log\xi + \left\{\begin{aligned}&\frac{(-n)(n+1)}{1^2}\left(\frac{1}{-n}+\frac{1}{n+1}-\frac{2}{1}\right)\xi\\&+\frac{(-n)(-n+1)(n+1)(n+2)}{1^2 \cdot 2^2}\\&\qquad\times\left(\begin{aligned}&\frac{1}{-n}+\frac{1}{-n+1}+\frac{1}{n+1}\\&+\frac{1}{n+2}-\frac{2}{1}-\frac{2}{2}\end{aligned}\right)\xi^2+\ \ldots\end{aligned}\right\}$$

The second solution $Q_n(x)$ of Legendre's Equation is then of the form

$$Q_n(x) = Ay_2 + By_1,$$

and since it is linearly independent of $P_n(x)$, or y_1, A cannot be zero : thus the required result is obtained.

10. If n is not an integer, show that, for $|x+1| < 2$,

$$P_n(x) = \frac{\sin n\pi}{\pi}\log\left(\frac{1+x}{2}\right)F\left(-n,\ n+1,\ 1,\ \frac{1+x}{2}\right) + R_n\left(\frac{1+x}{2}\right),$$

where $R_n\left(\frac{1+x}{2}\right)$ is a series of powers of $\left(\frac{1+x}{2}\right)$.

It can be proved (Bromwich, *Infinite Series*, Second Edition, p. 149) that, if the series Σa_n, Σb_n are both divergent, while the series $\Sigma a_n x^n$, $\Sigma b_n x^n$ are convergent for $|x| < 1$, and if a_n/b_n tends to a definite limit when n tends to infinity,

$$\operatorname*{Lim}_{x\to 1}\frac{\Sigma a_n x^n}{\Sigma b_n x^n} = \operatorname*{Lim}_{n\to\infty}\frac{a_n}{b_n}.$$

Now let $\Sigma a_n\xi^n$ and $\Sigma b_n\xi^n$ denote the series (Cf. 7),

$$P'_n(x) = -\frac{(-n)(n+1)}{1!}\tfrac{1}{2} - \frac{(-n)(-n+1)(n+1)(n+2)}{2!\,1!}\tfrac{1}{2}\xi - \ldots$$

and $$\frac{1}{1+x} = \frac{1}{2-2\xi} = \tfrac{1}{2}(1+\xi+\xi^2+\ldots),$$

where $\xi = (1-x)/2$; both series are divergent when $\xi = 1$ or $x = -1$: then

$$\frac{a_r}{b_r} = -\frac{(-n)(-n+1)\ldots(-n+r-1)(n+1)(n+2)\ldots(n+r)}{r!\,(r-1)!}$$
$$= -\frac{(-n)(-n+1)\ldots(-n+r-1)}{(r-1)!\,r^{-n}}\,\frac{(n+1)(n+2)\ldots(n+1+r-1)}{(r-1)!\,r^{n+1}},$$

so that, from (IV., 34),

$$\operatorname*{Lim}_{r\to\infty}\frac{a_r}{b_r} = -\frac{1}{\Gamma(-n)\Gamma(n+1)} = \frac{\sin n\pi}{\pi},$$

by (IV., 40). Hence

$$\operatorname*{Lim}_{x\to-1}(1+x)P'_n(x) = \frac{\sin n\pi}{\pi}.$$

If now the substitution $x = 2\xi - 1$ be made in Legendre's Equation, it is transformed into the equation discussed in *ex.* 9; namely

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + (1-2\xi)\frac{dy}{d\xi} + n(n+1)y = 0,$$

and the general solution is

$$y = A\log\xi F(-n, n+1, 1, \xi) + R_n(\xi),$$

where $R_n(\xi)$ is a convergent series. Thus

$$P_n(x) = A\log\left(\frac{1+x}{2}\right)F\left(-n, n+1, 1, \frac{1+x}{2}\right) + R_n\left(\frac{1+x}{2}\right)$$

and

$$(1+x)P'_n(x) = AF\left(-n, n+1, 1, \frac{1+x}{2}\right) + (1+x)\log(1+x)S_n\left(\frac{1+x}{2}\right),$$
$$+ (1+x)T_n\left(\frac{1+x}{2}\right),$$

where $S_n\left(\frac{1+x}{2}\right)$ and $T_n\left(\frac{1+x}{2}\right)$ are convergent series. Hence

$$\operatorname*{Lim}_{x\to-1}(1+x)P'_n(x) = A,$$

so that $A = \dfrac{\sin n\pi}{\pi}$. The required expression for $P_n(x)$ has therefore been established.

11. If n is zero or a positive integer, show that neither $Q_n(0)$ nor $Q'_n(0)$ has the value zero.

From *exs.* 7, 8, if n is even, $Q_n(0) = \tfrac{1}{2}P_n(0)\log(-1)$, which has a non-zero imaginary value. Again, (§ 5, Cor.),

$$Q_n(x)P'_n(x) - P_n(x)Q'_n(x) = \frac{1}{x^2 - 1}, \quad . \quad . \quad . \quad (a)$$

so that, if n is odd, $Q_n(0)P'_n(0) = -1$, and therefore $Q_n(0)$ does not vanish. Thus $Q_n(0)$ is non-zero for $n = 0, 1, 2, 3, \ldots$

Again, from (*a*), if n is even $P_n(0)Q'_n(0) = 1$, so that $Q'_n(0)$ does not vanish. If n is odd, on differentiating the formula of *ex.* 7, it is found that $Q'_n(0) = \frac{1}{2}P'_n(0) \log(-1)$; but $\log(-1)$ is imaginary, and therefore $Q'_n(0)$ cannot be zero. Thus $Q'_n(0)$ is non-zero for $n = 0, 1, 2, 3, \ldots$

CHAPER VII

THE ASSOCIATED LEGENDRE FUNCTIONS OF INTEGRAL ORDER

§ 1. Solution of Legendre's Associated Equation. In the equation (IV., 13)

$$(1 - x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1 - x^2}\right\}y = 0 \qquad (1)$$

put $y = (x^2 - 1)^{\frac{1}{2}m}u$; then

$$(1 - x^2)u'' - 2(m+1)xu' + (n - m)(n + m + 1)u = 0. \quad (2)$$

Again, by differentiating Legendre's equation (IV., 12)

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

m times, we can show that

$$(1 - x^2)\frac{d^{m+2}y}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}y}{dx^{m+1}} + (n - m)(n + m + 1)\frac{d^m y}{dx^m} = 0, \qquad (2a)$$

which is the same equation as (2).

Accordingly, if m is a positive integer, two independent solutions of Legendre's Associated Equation are

$$P_n^m(x) = (x^2 - 1)^{\frac{1}{2}m}\frac{d^m P_n(x)}{dx^m}, \quad . \quad . \quad . \quad (3)$$

and

$$Q_n^m(x) = (x^2 - 1)^{\frac{1}{2}m}\frac{d^m Q_n(x)}{dx^m}. \quad . \quad . \quad . \quad (4)$$

These functions $P_n^m(x)$ and $Q_n^m(x)$ are known as the *Associated Legendre Functions, of degree* n *and order* m, *of the First and Second Kinds respectively.* The amplitude of $(x^2 - 1)$ is taken to be zero when x is real and greater than 1.

Since (VI., 7)

$$P_n(x) = \sum_{r=0}^{\infty} \frac{\Gamma(n + r + 1)}{\Gamma(n - r + 1)(r!)^2}\left(\frac{x - 1}{2}\right)^r,$$

$$\frac{d^m P_n(x)}{dx^m} = \frac{1}{2^m}\sum_{r=m}^{\infty}\frac{\Gamma(n+r+1)}{\Gamma(n-r+1)r!\,(r-m)!}\left(\frac{x-1}{2}\right)^{r-m}$$

$$= \frac{1}{2^m}\sum_{r=0}^{\infty}\frac{\Gamma(n+m+r+1)}{\Gamma(n-m-r+1)(m+r)!\,r!}\left(\frac{x-1}{2}\right)^{r}$$

$$= \frac{1}{2^m}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\,.\,m!}$$

$$\times\ \mathrm{F}\left(n+m+1,\ m-n,\ m+1,\ \frac{1-x}{2}\right).$$

Hence

$$\mathrm{P}_n^m(x) = \frac{\Gamma(n+m+1)}{2^m\,.\,m!\,\Gamma(n-m+1)}(x^2-1)^{\frac{1}{2}m}$$

$$\times\ \mathrm{F}\left(m-n,\ m+n+1,\ m+1,\ \frac{1-x}{2}\right) \quad (5)$$

$$= \frac{\Gamma(n+m+1)}{m!\,\Gamma(n-m+1)}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}$$

$$\times\ \mathrm{F}\left(-n,\ n+1,\ m+1,\ \frac{1-x}{2}\right) \quad (6)$$

by (IV., 28)

Note—If n is zero or a positive integer, and $m > n$, $\mathrm{P}_n^m(x) = 0$, since $1/\Gamma(n-m+1) = 0$ This also follows from (3), since, when n is a positive integer, $\mathrm{P}_n(x)$ is a polynomial of degree n

Again, since (VI., 6)

$$\mathrm{Q}_n(x) = \frac{\sqrt{\pi}}{2^{n+1}}\sum_{r=0}^{\infty}\frac{\Gamma(n+2r+1)}{r!\,\Gamma(n+r+\frac{3}{2})}\frac{1}{2^{2r}x^{n+2r+1}},$$

$$\frac{d^m \mathrm{Q}_n(x)}{dx^m} = (-1)^m\frac{\sqrt{\pi}}{2^{n+1}}\sum_{r=0}^{\infty}\frac{\Gamma(n+m+2r+1)}{r!\,\Gamma(n+r+\frac{3}{2})}\frac{1}{2^{2r}x^{n+m+2r+1}},$$

so that

$$\mathrm{Q}_n^m(x) = \frac{(-1)^m(x^2-1)^{\frac{1}{2}m}}{2^{n+1}}\cdot\frac{\sqrt{\pi}\,.\,\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})}\cdot\frac{1}{x^{n+m+1}}$$

$$\times\ \mathrm{F}\left(\frac{n+m+1}{2},\ \frac{n+m+2}{2},\ n+\tfrac{3}{2},\ \frac{1}{x^2}\right). \quad (7)$$

Next, in Legendre's Associated Equation (1) make the substitution $y = (x^2 - 1)^{-\frac{1}{2}m}u$; it becomes

$$(1 - x^2)u'' - 2(1 - m)xu' + (n + m)(n - m + 1)u = 0. \quad (8)$$

Now differentiate this equation m times with regard to x; then

$$(1 - x^2)\frac{d^{m+2}u}{dx^{m+2}} - 2x\frac{d^{m+1}u}{dx^{m+1}} + n(n + 1)\frac{d^m u}{dx^m} = 0,$$

so that the mth derivative of an integral of (8) satisfies Legendre's Equation. Therefore, if m is a positive integer, the function

$$\int_1^x\int_1^\xi\int_1^\xi \cdots \int_1^\xi P_n(\xi)(d\xi)^m,$$

satisfies (8), and, if m is a positive integer such that $m < R(n + 1)$, the function

$$\int_\infty^x\int_\infty^\xi\int_\infty^\xi \cdots \int_\infty^\xi Q_n(\xi)(d\xi)^m$$

satisfies (8). Thus two new solutions $P_n^{-m}(x)$ and $Q_n^{-m}(x)$ of Legendre's Associated Equation can be defined, subject to these limitations on m, by the equations

$$P_n^{-m}(x) = (x^2 - 1)^{-\frac{1}{2}m}\int_1^x\int_1^\xi \cdots \int_1^\xi P_n(\xi)(d\xi)^m, \quad (9)$$

and $$Q_n^{-m}(x) = (x^2 - 1)^{-\frac{1}{2}m}\int_\infty^x\int_\infty^\xi \cdots \int_\infty^\xi Q_n(\xi)(d\xi)^m. \quad (10)$$

Example. Show that

(i) $P_n^{m+2}(x) + \frac{2(m + 1)x}{\sqrt{(x^2 - 1)}}P_n^{m+1}(x) = (n - m)(n + m + 1)P_n^m(x)$,

(ii) $Q_n^{m+2}(x) + \frac{2(m + 1)x}{\sqrt{(x^2 - 1)}}Q_n^{m+1}(x) = (n - m)(n + m + 1)Q_n^m(x)$.

[Compare formulæ (3), (4), and (2a).]

§ 2. **Relations between the Associated Legendre Functions.** Since the four functions $P_n^m(x)$, $P_n^{-m}(x)$, $Q_n^m(x)$, $Q_n^{-m}(x)$ satisfy the same differential equation (1), they cannot all be independent. The relations connecting them can be found as follows. From (VI., 7),

$$P_n(x) = \sum_{r=0}^{\infty} \frac{\Gamma(n + r + 1)}{\Gamma(n - r + 1)(r!)^2}\left(\frac{x - 1}{2}\right)^r,$$

so that

$$\int_1^x\int_1^\xi\int_1^\xi\cdots\int_1^\xi P_n(\xi)(d\xi)^m$$

$$= 2^m\sum_{r=0}^{\infty}\frac{\Gamma(n+r+1)}{\Gamma(n-r+1)\,.\,r\,!\,(m+r)\,!}\left(\frac{x-1}{2}\right)^{m+r}$$

$$= \frac{1}{m\,!}(x-1)^m F\left(-n, n+1, m+1, \frac{1-x}{2}\right).$$

Thus

$$P_n^{-m}(x) = \frac{1}{m\,!}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} F\left(-n, n+1, m+1, \frac{1-x}{2}\right) \qquad (11)$$

$$= \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}P_n^m(x) \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

by (6).

If m and n are positive integers such that $m < n$, this relation can also be deduced from (V., 13).

Again, from (VI., 6a) and (10), if m is a positive integer less than $n + 1$,

$$Q_n^{-m}(x) = (-1)^m(x^2-1)^{-\frac{1}{2}m}\frac{\sqrt{\pi}}{2^{n+1}}$$

$$\sum_{r=0}^{\infty}\frac{\Gamma(n-m+2r+1)}{r\,!\,\Gamma(n+\frac{1}{2}+r+1)}\frac{1}{2^{2r}x^{n-m+2r+1}}$$

$$=(-1)^m\frac{\sqrt{\pi}}{2^{n+1}}\frac{\Gamma(n-m+1)}{\Gamma(n+\frac{3}{2})}(x^2-1)^{-\frac{1}{2}m}\frac{1}{x^{n-m+1}}$$

$$\times\ F\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\tfrac{3}{2}, \frac{1}{x^2}\right)$$

$$=(-1)^m\frac{\sqrt{\pi}}{2^{n+1}}\frac{\Gamma(n-m+1)}{\Gamma(n+\frac{3}{2})}(x^2-1)^{-\frac{1}{2}m}\frac{1}{x^{n-m+1}}$$

$$\times\ \left(1-\frac{1}{x^2}\right)^m F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}, n+\tfrac{3}{2}, \frac{1}{x^2}\right), \text{ by (IV., 28)}$$

$$= \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}Q_n^m(x), \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

by (7).

§ 3. **Ferrers' Associated Legendre Function of the First Kind.** When x is real and $-1 < x < 1$, it is sometimes convenient to use Ferrers' function

$$T_n^m(x) = (1-x^2)^{\frac{1}{2}m}\frac{d^m}{dx^m}P_n(x). \quad . \quad . \qquad (14)$$

instead of $P_n^m(x)$. If x, starting from a point on the real axis to the right of $x = 1$, be made to pass round $x = 1$ in the positive direction to a point on the real axis between -1 and $+1$, the function $(x^2 - 1)^{\frac{1}{2}m}$ comes to have the value $(1 - x^2)^{\frac{1}{2}m}e^{\frac{1}{2}m\pi i}$, and

$$P_n^m(x) = e^{\frac{1}{2}m\pi i}T_n^m(x). \quad . \quad . \quad . \quad (15)$$

On the other hand, if x passes round $x = 1$ in the negative direction from its original to its final position,

$$P_n^m(x) = e^{-\frac{1}{2}m\pi i}T_n^m(x). \quad . \quad . \quad . \quad (15')$$

Example. Show that, if m and n are positive integers, and m . n,

$$T_n^m(\mu) = \frac{(2n)!}{2^n . n! (n - m)!} \sin^m\theta$$
$$\times \left\{ \begin{array}{l} \mu^{n-m} - \dfrac{(n - m)(n - m - 1)}{2(2n - 1)}\mu^{n-m-2} \\ \quad + \dfrac{(n - m)(n - m - 1)(n - m - 2)(n - m - 3)}{2 . 4(2n - 1)(2n - 3)}\mu^{n-m-4} - \dots \end{array} \right\}.$$

[Cf. the example at the end of § 2, Ch. V.]

§ 4. **Integrals of Products of Associated Legendre Functions.** It will now be shown that, if m, n, and k are positive integers such that $k \not> m$, $k \not> n$,

$$\int_{-1}^{1} P_m^k(x)P_n^k(x)dx = 0, \; m \neq n, \quad . \quad . \quad (16)$$

$$\int_{-1}^{1} \left\{P_n^k(x)\right\}^2 dx = (-1)^k\frac{(n + k)!}{(n - k)!}\frac{2}{2n + 1}. \quad (17)$$

From (1) it follows that

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_n^k(x)}{dx}\right\} + \left\{n(n + 1) - \frac{k^2}{1 - x^2}\right\}P_n^k(x) = 0,$$

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_m^k(x)}{dx}\right\} + \left\{m(m + 1) - \frac{k^2}{1 - x^2}\right\}P_m^k(x) = 0.$$

Multiply the first equation by $P_m^k(x)$, the second by $P_n^k(x)$, subtract, and integrate between the limits -1 and $+1$; thus

$$(n - m)(n + m + 1)\int_{-1}^{1} P_m^k(x)P_n^k(x)dx$$
$$= \int_{-1}^{1}\left[P_n^k(x)\frac{d}{dx}\left\{(1 - x^2)\frac{dP_m^k(x)}{dx}\right\}\right.$$
$$\left. - P_m^k(x)\frac{d}{dx}\left\{(1 - x^2)\frac{dP_n^k(x)}{dx}\right\}\right]dx$$

$$= \int_{-1}^{1} \frac{d}{dx}\left[(1 - x^2)\left\{ P_n^k(x)\frac{d}{dx}P_m^k(x) - P_m^k(x)\frac{d}{dx}P_n^k(x)\right\}\right]dx = 0,$$

from which, if $m \neq n$, the first equation follows.

Again, from (12),

$$\int_{-1}^{1}\{P_n^k(x)\}^2 dx = \frac{(n+k)!}{(n-k)!}\int_{-1}^{1} P_n^k(x)P_n^{-k}(x)dx$$

$$= \frac{(n+k)!}{2^{2n}.(n!)^2.(n-k)!}\int_{-1}^{1}\frac{d^{n+k}}{dx^{n+k}}(x^2-1)^n\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n dx,$$

by (3) and (9). If the integral on the right be integrated by parts k times, the equation becomes

$$\int_{-1}^{1}\{P_n^k(x)\}^2 dx = \frac{(n+k)!}{2^{2n}.(n!)^2.(n-k)!}(-1)^k\int_{-1}^{1}\left\{\frac{d^n(x^2-1)^n}{dx^n}\right\}^2 dx$$

$$= (-1)^k\frac{(n+k)!}{(n-k)!}\int_{-1}^{1}\{P_n(x)\}^2 dx$$

$$= (-1)^k\frac{(n+k)!}{(n-k)!}\cdot\frac{2}{2n+1}.$$

Corollary. $$\int_{-1}^{1}\{T_n^k(x)\}^2 dx = \frac{(n+k)!}{(n-k)!}\cdot\frac{2}{2n+1}. \qquad (18)$$

§ 5. **Laplace's First Integral for $P_n^m(x)$.** If n and m are positive integers ($m \leq n$),

$$P_n^m(x) = \frac{(n+m)!}{n!}\frac{1}{\pi}\int_0^\pi\{x + \sqrt{(x^2-1)}\cos\phi\}^n\cos m\phi\, d\phi. \qquad (19)$$

This can be proved as follows. Expand the function $\{(x+y)^2 - 1\}^n$ in powers of y by means of Taylor's Theorem; thus

$$\{(x+y)^2 - 1\}^n = (x^2-1)^n + \sum_{r=1}^{2n}\frac{y^r}{r!}\frac{d^r}{dx^r}(x^2-1)^n.$$

Now divide by y^n and rearrange the series as follows :—

$$\frac{\{(x+y)^2-1\}^n}{y^n} = \frac{1}{n!}\frac{d^n}{dx^n}(x^2-1)^n + \sum_{m=1}^{n}\frac{y^m}{(n+m)!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$$

$$+ \sum_{m=1}^{n}\frac{y^{-m}}{(n-m)!}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n.$$

In this equation put $y = \sqrt{(x^2 - 1)} . e^{i\phi}$ and divide by 2^n; this gives

$$\{x + \sqrt{(x^2 - 1)}\cos\phi\}^n = \frac{1}{2^n . n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$
$$+ \sum_{m=1}^{n} \frac{\cos m\phi + i \sin m\phi}{2^n . (n+m)!}(x^2 - 1)^{\frac{1}{2}m}\frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n$$
$$+ \sum_{m=1}^{n} \frac{\cos m\phi - i \sin m\phi}{2^n . (n-m)!}(x^2 - 1)^{-\frac{1}{2}m}\frac{d^{n-m}}{dx^{n-m}}(x^2 - 1)^n.$$

If x is real and greater than 1, the expression on the left is real; hence the imaginary part of the right-hand side must vanish; *i.e.*,

$$\sum_{m=1}^{n} \sin m\phi \left\{ \begin{array}{l} \frac{1}{(n+m)!}(x^2 - 1)^{\frac{1}{2}m}\frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n \\ - \frac{1}{(n-m)!}(x^2 - 1)^{-\frac{1}{2}m}\frac{d^{n-m}}{dx^{n-m}}(x^2 - 1)^n \end{array} \right\} = 0.$$

But this equation holds for all values of ϕ, so that the coefficients of $\sin m\phi$ $(m = 1, 2, \ldots, n)$ must all be zero. Accordingly

$$\frac{(x^2 - 1)^{\frac{1}{2}m}}{(n+m)!}\frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n = \frac{(x^2 - 1)^{-\frac{1}{2}m}}{(n-m)!}\frac{d^{n-m}}{dx^{n-m}}(x^2 - 1)^n,$$

which is (V., 13) and a particular case of the formula (12).

Thus

$$\{x + \sqrt{(x^2 - 1)}\cos\phi\}^n = \frac{1}{2^n . n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$
$$+ 2\sum_{m=1}^{n} \cos m\phi \frac{(x^2 - 1)^{\frac{1}{2}m}}{2^n . (n+m)!}\frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n$$
$$= P_n(x) + 2\sum_{m=1}^{n} \frac{n!}{(n+m)!}\cos m\phi \, P_n^m(x). \qquad (20)$$

To obtain the formula (19) it is only necessary to multiply this equation by $\cos m\phi$ and to integrate from 0 to π.

Corollary. By writing $\pi - \phi$ for ϕ in (19), we deduce the formula

$$P_n^m(x) = (-1)^m \frac{(n+m)!}{n!}\frac{1}{\pi}\int_0^\pi \{x - \sqrt{(x^2 - 1)}\cos\phi\}^n \cos m\phi \, d\phi. \qquad (19')$$

Example. Show that, if n is a positive integer, and x, y, z are rectangular co-ordinates,

$$(z + ix\cos\alpha + iy\sin\alpha)^n$$
$$= r^n\left\{\mathrm{P}_n(\mu) + 2\sum_{m=1}^{n}\frac{n!}{(n+m)!}\mathrm{P}_n^m(\mu)\cos m(\phi - \alpha)\right\}.$$

§ 6. **Laplace's Second Integral for $\mathrm{P}_n^{-m}(x)$.** If n and m are positive integers

$$\mathrm{P}_n^{-m}(x) = (-1)^m\frac{n!}{(n+m)!}\frac{1}{\pi}\int_0^\pi\frac{\cos m\phi\, d\phi}{\{x + \sqrt{(x^2-1)}\cos\phi\}^{n+1}}. \qquad (21)$$

In order to establish this formula consider the function

$$\{(x+y)^2 - 1\}^{-n-1} = (x + y + 1)^{-n-1}(x + y - 1)^{-n-1},$$

where $y = \sqrt{(x^2 - 1)}e^{i\phi}$ and $\mathrm{R}(x) > 0$. Since $|e^{i\phi}| = 1$, $|y|$ lies between $|x - 1|$ and $|x + 1|$, so that one of the factors $(x + y + 1)^{-n-1}$, $(x + y - 1)^{-n-1}$ can be expanded in ascending, and the other in descending, powers of y. Thus we may write

$$(2y)^{n+1}\{(x+y)^2 - 1\}^{-n-1} = c_0 + \sum_{m=1}^{\infty}\{c_m y^m + c_{-m}y^{-m}\}. \qquad (22)$$

Now put $y = \sqrt{(x^2 - 1)}e^{i\phi}$, and (22) becomes

$$\{x + \sqrt{(x^2 - 1)}\cos\phi\}^{-n-1}$$
$$= c_0 + \sum_{m=1}^{\infty}\{c_m(x^2 - 1)^{\frac{1}{2}m} + c_{-m}(x^2 - 1)^{-\frac{1}{2}m}\}\cos m\phi$$
$$+ i\sum_{m=1}^{\infty}\{c_m(x^2 - 1)^{\frac{1}{2}m} - c_{-m}(x^2 - 1)^{-\frac{1}{2}m}\}\sin m\phi.$$

Since the left-hand side of this equation is unaltered when the sign of ϕ is changed, it follows that

$$\sum_{m=1}^{\infty}\{c_m(x^2 - 1)^{\frac{1}{2}m} - c_{-m}(x^2 - 1)^{-\frac{1}{2}m}\}\sin m\phi = 0,$$

and, since this equation holds for all values of ϕ, the coefficients of the sines must all vanish. Hence

$$c_m(x^2 - 1)^{\frac{1}{2}m} = c_{-m}(x^2 - 1)^{-\frac{1}{2}m}\ . \qquad . \quad (23)$$

Accordingly

$$\{x+\sqrt{(x^2-1)}\cos\phi\}^{-n-1}=c_0+2\sum_{m=1}^{\infty}c_{-m}(x^2-1)^{-\frac{1}{2}m}\cos m\phi. \tag{24}$$

Integrating this equation with regard to ϕ from 0 to π, we deduce from (V., 48) that

$$c_0=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^{-n-1}d\phi=P_n(x).$$

Again, from (22),

$$2^{n+1}\{(x+y)^2-1\}^{-n-1}$$
$$=c_0y^{-n-1}+\sum_{m=1}^{\infty}\{c_my^{m-n-1}+c_{-m}y^{-m-n-1}\}.$$

Now the left-hand side of this equation is symmetrical in x and y, so that its mth partial derivatives with regard to x and y are identical; thus

$$\frac{\partial^m c_0}{\partial x^m}y^{-n-1}+\sum_{m=1}^{\infty}\left\{\frac{\partial^m c_m}{\partial x^m}y^{m-n-1}+\frac{\partial^m c_{-m}}{\partial x^m}y^{-m-n-1}\right\}$$
$$=\frac{\partial^m}{\partial y^m}\left[c_0y^{-n-1}+\sum_{m=1}^{\infty}\{c_my^{m-n-1}+c_{-m}y^{-m-n-1}\}\right].$$

By equating the coefficients of y^{-m-n-1}, we find that

$$\frac{\partial^m}{\partial x^m}c_{-m}=(-1)^m c_0\frac{(n+m)!}{n!}=(-1)^m\frac{(n+m)!}{n!}P_n(x). \tag{25}$$

Now, if $x=1$, the left-hand side of (22) becomes $(1+\frac{1}{2}y)^{-n-1}$, and $\frac{1}{2}y<1$, so that c_m is finite and c_{-m} is zero. Hence, from (23), $c_{-m}=(x^2-1)^m c_m$, where c_m remains finite when x tends to 1. Therefore, integrating (25), we get

$$c_{-m}=(-1)^m\frac{(n+m)!}{n!}\int_1^x\int_1^{\xi}\int_1^{\xi}\cdots\int_1^{\xi}P_n(\xi)(d\xi)^m$$
$$=(-1)^m\frac{(n+m)!}{n!}(x^2-1)^{\frac{1}{2}m}P_n^{-m}(x),$$

by (9). Thus

$$\{x+\sqrt{(x^2-1)}\cos\phi\}^{-n-1}$$
$$=P_n(x)+2\sum_{m=1}^{\infty}(-1)^m\frac{(n+m)!}{n!}\cos m\phi\, P_n^{-m}(x). \tag{26}$$

If this equation is multiplied by $\cos m\phi$ and integrated from 0 to π, the formula (21) is obtained.

Corollary 1. By writing $\pi - \phi$ for ϕ in (21), we deduce that

$$\mathrm{P}_n^{-m}(x) = \frac{n\,!}{(n+m)\,!}\,\frac{1}{\pi}\int_0^\pi \frac{\cos m\phi\, d\phi}{\{x - \sqrt{(x^2 - 1)}\cos\phi\}^{n+1}}. \quad (21')$$

Corollary 2. From (12) it follows that, if $m \leq n$,

$$\begin{aligned}\mathrm{P}_n^m(x) &= (-1)^m \frac{n\,!}{(n-m)\,!}\,\frac{1}{\pi}\int_0^\pi \frac{\cos m\phi\, d\phi}{\{x + \sqrt{(x^2 - 1)}\cos\phi\}^{n+1}} \\ &= \frac{n\,!}{(n-m)\,!}\,\frac{1}{\pi}\int_0^\pi \frac{\cos m\phi\, d\phi}{\{x - \sqrt{(x^2 - 1)}\cos\phi\}^{n+1}}. \quad (27)\end{aligned}$$

§ 7. **Spherical Harmonics of Integral Degree.** From Chapter IV., § 6, it follows that, if n and m are positive integers ($m \leq n$),

$$(\mathrm{A}\cos m\phi + \mathrm{B}\sin m\phi)\mathrm{T}_n^m(\cos\theta)$$

is a surface spherical harmonic of degree n. If $m = 0$, the harmonic is simply the Legendre Coefficient $\mathrm{P}_n(\cos\theta)$. Now it has been shown (Chap. V., § 5) that $\mathrm{P}_n(\mu)$ has n distinct zeros between -1 and $+1$, arranged symmetrically about $\mu = 0$, so that $\mathrm{P}_n(\cos\theta)$ has n distinct zeros between 0 and π, arranged symmetrically about $\theta = \frac{1}{2}\pi$. Accordingly, on a sphere with the origin as centre, the function $\mathrm{P}_n(\cos\theta)$ vanishes on n circles (Fig. 14) with poles at the points $\theta = 0$ and $\theta = \pi$. The circles are symmetrically situated with respect to the great circle of which these points are the poles, and, if n is an odd number, this circle is itself one of the set. Similarly the locus of points on the sphere at which the function has a constant value consists of a number of parallel circles. Because of this division of the sphere into zones by sets of parallel circles the functions $\mathrm{P}_n(\cos\theta)$ are called *Zonal Harmonics*. The point $\theta = 0$ is called the *Pole*, and the diameter through the pole the *Axis* of the zonal harmonic.

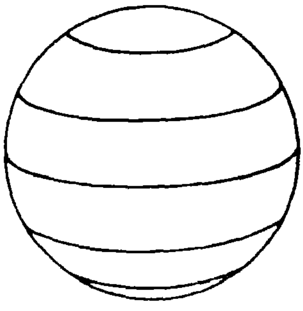

FIG. 14.

If $0 < m < n$ the spherical harmonic is of the form

$$(A \cos m\phi + B \sin m\phi) \sin^m \theta \frac{d^{n+m}}{d\mu^{n+m}}(\mu^2 - 1)^n.$$

The first factor vanishes when $A \cos m\phi + B \sin m\phi = 0$; *i.e.* when $\tan m\phi = - A/B$, and on the sphere (Fig. 15) this corresponds to m great circles through the pole $\theta = 0$, the angle between the planes of any two consecutive great circles being π/m. The second factor vanishes at the points $\theta = 0$ and $\theta = \pi$, and the third on $n - m$ circles with $\theta = 0$ as pole, arranged like the corresponding circles in the case of the zonal harmonics. Since the two sets of circles intersect orthogonally, these harmonics are called *Tesseral Harmonics*

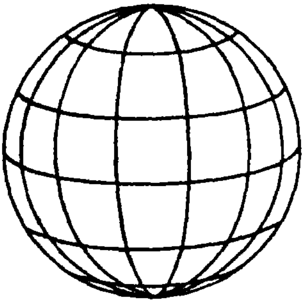

FIG. 15.

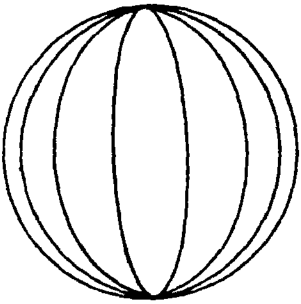

FIG. 16.

Again, if $m = n$, the spherical harmonic is of the form

$$(A \cos n\phi + B \sin n\phi) \sin^n \theta,$$

which vanishes when $\tan n\phi = - A/B$ or when $\theta = 0$ or π. This corresponds on the sphere (Fig. 16) to the points $\theta = 0$ and $\theta = \pi$, and to n great circles through these points, the angle between the planes of any two consecutive circles being π/n. As the sphere is thus divided up into $2n$ sectors, these functions are called *Sectorial Harmonics*.

The following list contains all the Surface Spherical Harmonics of degrees 1, 2, 3, and 4:—

$P_1(\cos\theta) = \cos\theta, \begin{matrix}\cos\phi\\ \sin\phi\end{matrix} T_1^1(\cos\theta) = \begin{matrix}\cos\phi\\ \sin\phi\end{matrix} \sin\theta:$

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1),$$

$$\begin{array}{c}\cos\phi\\ \sin\phi\end{array} T_2^1(\cos\theta) = \begin{array}{c}\cos\phi\\ \sin\phi\end{array} 3\sin\theta\cos\theta,$$

$$\begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} T_2^2(\cos\theta) = \begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} 3\sin^2\theta:$$

$$P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta),$$

$$\begin{array}{c}\cos\phi\\ \sin\phi\end{array} T_3^1(\cos\theta) = \begin{array}{c}\cos\phi\\ \sin\phi\end{array} \tfrac{3}{2}\sin\theta\,(5\cos^2\theta - 1),$$

$$\begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} T_3^2(\cos\theta) = \begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} 15\sin^2\theta\cos\theta,$$

$$\begin{array}{c}\cos 3\phi\\ \sin 3\phi\end{array} T_3^3(\cos\theta) = \begin{array}{c}\cos 3\phi\\ \sin 3\phi\end{array} 15\sin^3\theta:$$

$$P_4(\cos\theta) = \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3),$$

$$\begin{array}{c}\cos\phi\\ \sin\phi\end{array} T_4^1(\cos\theta) = \begin{array}{c}\cos\phi\\ \sin\phi\end{array} \tfrac{5}{2}\sin\theta\,(7\cos^3\theta - 3\cos\theta),$$

$$\begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} T_4^2(\cos\theta) = \begin{array}{c}\cos 2\phi\\ \sin 2\phi\end{array} \tfrac{15}{2}\sin^2\theta\,(7\cos^2\theta - 1),$$

$$\begin{array}{c}\cos 3\phi\\ \sin 3\phi\end{array} T_4^3(\cos\theta) = \begin{array}{c}\cos 3\phi\\ \sin 3\phi\end{array} 105\sin^3\theta\cos\theta,$$

$$\begin{array}{c}\cos 4\phi\\ \sin 4\phi\end{array} T_4^4(\cos\theta) = \begin{array}{c}\cos 4\phi\\ \sin 4\phi\end{array} 105\sin^4\theta.$$

§ 8. **Expression of a Surface Spherical Harmonic of integral degree [*] in terms of Zonal, Tesseral, and Sectorial Harmonics.** The general homogeneous polynomial $V_n(x, y, z)$ of degree n in x, y, z, where n is a positive integer, contains $\frac{1}{2}(n + 1)(n + 2)$ constants. If this function is a solid spherical harmonic it will satisfy Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Let $V_n(x, y, z)$ be substituted for V in the left-hand side of this equation; the resulting function is of degree $n - 2$, and therefore contains $\frac{1}{2}n(n - 1)$ terms. In order that it should satisfy the equation for all values of x, y, z, the coefficients of each of these terms must vanish. This gives $\frac{1}{2}n(n - 1)$

---

[*] The harmonic is supposed to be a homogeneous rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$.

relations which must hold between the $\frac{1}{2}(n+1)(n+2)$ constants if $V_n(x, y, z)$ is to be a solid spherical harmonic; so that these constants can be expressed linearly in terms of $\frac{1}{2}\{(n+1)(n+2) - n(n-1)\}$ or $2n+1$ of them. Hence $V_n(x, y, z)$ can be put in the form

$$V_n(x, y, z) = a_1 X_1 + a_2 X_2 + \ldots + a_{2n+1} X_{2n+1},$$

where $a_1, a_2, \ldots, a_{2n+1}$ are arbitrary constants. The functions $X_1, X_2, \ldots, X_{2n+1}$ are all solid spherical harmonics of degree n; this can be made clear by equating one of the a's to unity and the others to zero. Also, the functions $X_1, X_2, \ldots, X_{2n+1}$ are linearly independent, as each of them contains at least one term (the original coefficient of the corresponding constant) which is not contained by any of the others.

Example. If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is a spherical harmonic, prove that $a + b + c = 0$, and hence show that every spherical harmonic of the second degree can be expressed linearly in terms of the five harmonics $x^2 - z^2$, $y^2 - z^2$, yz, zx, xy.

Again, let $Y_1, Y_2, \ldots, Y_{2n+1}$ be the surface spherical harmonics which correspond respectively to $X_1, X_2, \ldots, X_{2n+1}$, and let $V_n(\theta, \phi)$ be any surface spherical harmonic of degree n; from the previous paragraph it follows that $V_n(\theta, \phi)$ can be expressed as a linear combination of $Y_1, Y_2, \ldots, Y_{2n+1}$. Also, let $Z_1, Z_2, \ldots, Z_{2n+1}$ be the $2n+1$ linearly independent surface harmonics of degree n given by

$$Z_1 = P_n(\cos\theta),\ Z_{1+m} = \cos m\phi\, T_n^m(\cos\theta),\ (m = 1, 2, \ldots, n),$$

$$Z_{n+1+m} = \sin m\phi\, T_n^m(\cos\theta),\ (m = 1, 2, \ldots, n).$$

Then

$$Z_r = a_{r,1} Y_1 + a_{r,2} Y_2 + \ldots + a_{r,2n+1} Y_{2n+1},$$

where $r = 1, 2, \ldots, 2n+1$.

But, since the $2n+1$ Z's are all linearly independent, these equations can be solved in the form

$$Y_r = b_{r,1} Z_1 + b_{r,2} Z_2 + \ldots + b_{r,2n+1} Z_{2n+1},$$

where $r = 1, 2, \ldots, 2n+1$.

Hence, finally, any surface spherical harmonic $V_n(\theta, \phi)$ of degree n can be expressed in the form

$$V_n(\theta, \phi) = AP_n(\cos\theta) + \sum_{m=1}^{n} \{A_m \cos m\phi + B_m \sin m\phi\} T_n^m(\cos\theta), \qquad (28)$$

where the A's and B's are constants.

Note.—Since the function $(z + ix)^n$ satisfies Laplace's Equation for all values of n, it is a solid harmonic of degree n, and therefore $r^{-n}(z + ix)^n$ or $\{\mu + \sqrt{(\mu^2 - 1)} \cos\phi\}^n$ is a surface spherical harmonic of degree n. It follows that, if n is a positive integer, the latter function can be expressed in the form (28); the actual expansion is given in formula (20).

§ 9. **Integrals of Products of Surface Spherical Harmonics.** Let $X_m(\theta, \phi)$ and $Y_n(\theta, \phi)$ be two surface spherical harmonics of integral degrees m and n respectively; then, if $m \neq n$,

$$\int_0^{2\pi}\int_{-1}^{1} X_m(\theta, \phi) Y_n(\theta, \phi) d\mu d\phi = 0, \quad . \quad . \quad (29)$$

the double integral being taken over the surface of the unit sphere.

For, from (28),

$$X_m(\theta, \phi) = BP_m(\mu) + \sum_{k=1}^{m} \{B_k \cos k\phi + B'_k \sin k\phi\} T_m^k(\mu),$$

$$Y_n(\theta, \phi) = CP_n(\mu) + \sum_{k=1}^{n} \{C_k \cos k\phi + C'_k \sin k\phi\} T_n^k(\mu).$$

When these values are substituted for X_m and Y_n in (29), it is found that the integral of every term in the product has the value zero. This follows, for terms in which the k's are equal, from (V., 17) and (16), and for terms in which the k's are unequal, from the formulæ (cf. Ch. I., § 1):

$$\int_0^{2\pi} \cos p\phi \, d\phi = 0; \quad \int_0^{2\pi} \sin p\phi \cos q\phi \, d\phi = 0;$$

$$\int_0^{2\pi} \cos p\phi \cos q\phi \, d\phi = 0, \ (p \neq q); \quad \int_0^{2\pi} \sin p\phi \sin q\phi \, d\phi = 0, (p \neq q);$$

where p and q are integers or zero.

Again, if $X_n(\theta, \phi)$ and $Y_n(\theta, \phi)$ are surface spherical harmonics of the same integral degree n,

$$X_n(\theta, \phi) = BP_n(\mu) + \sum_{m=1}^{n} \{B_m \cos m\phi + B'_m \sin m\phi\} T_n^m(\mu),$$

$$Y_n(\theta, \phi) = CP_n(\mu) + \sum_{m=1}^{n} \{C_m \cos m\phi + C'_m \sin m\phi\} T_n^m(\mu);$$

and hence, using the formulæ employed in the previous paragraph, and also (V., 18), (18), and

$$\int_0^{2\pi} \cos^2 p\phi \, d\phi = \pi, \qquad \int_0^{2\pi} \sin^2 p\phi \, d\phi = \pi,$$

we deduce that

$$\int_0^{2\pi}\int_{-1}^{1} X_n(\theta, \phi) Y_n(\theta, \phi) d\mu d\phi$$

$$= \frac{2\pi}{2n+1}\left\{2BC + \sum_{m=1}^{n} (B_m C_m + B'_m C'_m)\frac{(n+m)!}{(n-m)!}\right\}. \quad (30)$$

In particular, if X_n is the zonal harmonic $P_n(\mu)$,

$$\int_0^{2\pi}\int_{-1}^{1} Y_n(\theta, \phi) P_n(\mu) d\mu d\phi = \frac{4\pi}{2n+1} C.$$

But, if $\theta = 0$, $\mu = 1$, and $T_n^m(\mu) = 0$, $(m = 1, 2, \ldots, n)$, since $(1 - \mu^2)^{\frac{1}{2}m}$ is a factor of $T_n^m(\mu)$. Thus the value of $Y_n(\theta, \phi)$ at the pole of $P_n(\mu)$ is C, which we may denote by $Y_n(1)$. Accordingly

$$\int_0^{2\pi}\int_{-1}^{1} Y_n(\theta, \phi) P_n(\mu) d\mu d\phi = \frac{4\pi}{2n+1} Y_n(1). \quad (31)$$

§ 10. **The Laplace's Coefficients.** The Laplace's Coefficients $P_n(\cos\gamma)$, where $n = 0, 1, 2, \ldots$ and (cf. Ch. IV., § 3)

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \quad (32)$$

are functions of the two variables θ and ϕ; they are, in fact, surface spherical harmonics in θ and ϕ. On a sphere with the origin as centre the arc of a great circle joining the variable point (θ, ϕ) and the fixed point (θ', ϕ') subtends the angle γ at the origin: and so, by analogy with the ordinary zonal harmonics, the Laplace's Coefficients are called *General Zonal Harmonics;* the point (θ', ϕ') is the *Pole*, and the diameter through the pole is the *Axis*, of the system.

Now, consider the integral

$$\int_0^{2\pi}\int_{-1}^{1} Y_n(\theta, \phi)P_n(\cos\gamma)\,d\mu\,d\phi.$$

Let the axes of co-ordinates be changed so that the new z-axis passes through the point (θ', ϕ') and let the new angular coordinates be Θ and Φ, so that $\Theta = \gamma$; also let $Y_n(\theta, \phi)$ become $X_n(\Theta, \Phi)$. Then the integral is equal to

$$\int_0^{2\pi}\int_{-1}^{1} X_n(\Theta, \Phi)P_n(\cos\Theta)d(\cos\Theta)d\Phi,$$

which, by (31), has the value $\frac{4\pi}{2n+1}X_n(1)$, or, what is the same thing, $\frac{4\pi}{2n+1}Y_n(\theta', \phi')$. Hence

$$\int_0^{2\pi}\int_{-1}^{1} Y_n(\theta, \phi)P_n(\cos\gamma)d\mu d\phi = \frac{4\pi}{2n+1}Y_n(\theta', \phi'). \tag{33}$$

§ 11. **The Addition Theorem for the Zonal Harmonics.** From (28), if n is a positive integer, and $\cos\gamma$ is given by (32),

$$P_n(\cos\gamma) = AP_n(\cos\theta)$$
$$+\sum_{m=1}^{n}\{A_m\cos m\phi + B_m\sin m\phi\}T_n^m(\cos\theta).$$

To determine the value of A_m multiply this equation by the surface harmonic $\cos m\phi T_n^m(\cos\theta)$ and integrate over the surface of the unit sphere; this gives, with the help of (33) and (30),

$$\frac{4\pi}{2n+1}\cos m\phi'\,T_n^m(\cos\theta') = \frac{2\pi}{2n+1}A_m\frac{(n+m)!}{(n-m)!},$$

so that

$$A_m = 2\frac{(n-m)!}{(n+m)!}\cos m\phi'\,T_n^m(\cos\theta').$$

Similarly

$$B_m = 2\frac{(n-m)!}{(n+m)!}\sin m\phi'\,T_n^m(\cos\theta'),$$

and

$$A = P_n(\cos\theta').$$

Thus we obtain the addition theorem

$$P_n(\cos\gamma) = P_n(\cos\theta)P_n(\cos\theta')$$
$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\cos m(\phi - \phi')T_n^m(\cos\theta)T_n^m(\cos\theta'). \quad (34)$$

If x and x' are written in place of $\cos\theta$ and $\cos\theta'$ respectively, equation (34) may be put in the form

$$P_n\{xx' - (x^2 - 1)^{\frac{1}{2}}(x'^2 - 1)^{\frac{1}{2}}\cos(\phi - \phi')\} = P_n(x)P_n(x')$$
$$+ 2\sum_{m=1}^{n}(-1)^m\frac{(n-m)!}{(n+m)!}\cos m(\phi - \phi')P_n^m(x)P_n^m(x'); \quad (34')$$

this formula is then valid for all values of x such that

$$-\pi < \text{amp}\,(x^2 - 1)^{\frac{1}{2}} \leqq \pi, \quad -\pi < \text{amp}\,(x'^2 - 1)^{\frac{1}{2}} \leqq \pi.$$

Corollary. $\int_0^{2\pi} P_n(\cos\gamma)d\phi = 2\pi P_n(\cos\theta)P_n(\cos\theta')$

§ 12. **Expansion of a rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, in terms of Surface Spherical Harmonics.** It will now be shown that, if the function $f(\theta, \phi)$ is a rational integral function of the co-ordinates of a point on the unit sphere (*i.e.* of $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$), it can be expressed as the sum of a finite number of spherical harmonics.

Any term of $f(\theta, \phi)$ is of the form

$$(\cos\theta)^{n-p}\sin^p\theta\,(\cos\phi)^{p-q}\sin^q\phi \quad . \quad . \quad (35)$$

multiplied by a constant, n, p, q being positive integers such that $p \leqq n$, $q \leqq p$. The cases of q even and q odd will be considered separately.

Case I. Let q be even: then

$$(\cos\phi)^{p-q}\sin^q\phi = (\cos\phi)^{p-q}(1 - \cos^2\phi)^{\frac{1}{2}q}$$

can be put in the form

$$\sum_k A_k\cos(p - 2k)\phi,$$

where $k = 0, 1, 2, \ldots$ up to $\frac{1}{2}p$ or $\frac{1}{2}(p - 1)$ according as p is even or odd. Thus (35) is equal to

$$(\cos\theta)^{n-p}(\sin\theta)^p\sum_k A_k\cos(p - 2k)\phi. \quad . \quad . \quad (36)$$

In order to obtain spherical harmonics of the type $\cos m\phi \sin^r \theta \dfrac{d^r}{d\mu^r} P_n(\mu)$ we next put the general term of (36) in the form

$$(\cos\theta)^{n-p}(1 - \cos^2\theta)^k (\sin\theta)^{p-2k} \cos(p - 2k)\phi,$$

or, if $m = p - 2k$,

$$\sin^m\theta \cos m\phi\, (\cos\theta)^{n-m-2k}(1 - \cos^2\theta)^k,$$

where $m + 2k \leqq n$. By expanding $(1 - \cos^2\theta)^k$ this can be written

$$\sin^m\theta \cos m\phi \{B_0(\cos\theta)^{n-m} + B_1(\cos\theta)^{n-m-2} + \dots \\ + B_k(\cos\theta)^{n-m-2k}\}. \quad (37)$$

Now, if $\mu = \cos\theta$,

$$\mu^{n-m-2r} = \frac{1}{(n-2r)(n-2r-1)\,.\;.\,(n-2r-m+1)} \frac{d^m}{d\mu^m}\mu^{n-2r};$$

but μ^{n-2r} can be expressed in the form

$$C_0 P_{n-2r}(\mu) + C_2 P_{n-2r-2}(\mu) + \dots,$$

the last term being a constant multiple of $P_1(\mu)$ or $P_0(\mu)$ according as n is odd or even. Hence μ^{n-m-2r} can be written

$$D_0 \frac{d^m}{d\mu^m} P_{n-2r}(\mu) + D_2 \frac{d^m}{d\mu^m} P_{n-2r-2}(\mu) + \dots,$$

and therefore

$$\sin^m\theta\,(\cos\theta)^{n-m-2r} \\ = K_0 P^m_{n-2r}(\cos\theta) + K_2 P^m_{n-2r-2}(\cos\theta) + \dots$$

It follows that (37) and therefore (35) can be expanded as a sum of surface spherical harmonics of the form

$$\cos m\phi\; P^m_{n-2r}(\cos\theta),$$

each multiplied by a constant, where $p - m$ is zero or an even integer.

Case II. Let q be odd; then

$$(\cos\phi)^{p-q}\sin^q\phi = (\cos\phi)^{p-q}(1 - \cos^2\phi)^{\frac{1}{2}(q-1)}\sin\phi,$$

and this can be written in the form

$$\sum_k A_k \sin(p - 2k)\phi.$$

In the same way as before it can now be shown that (35) is a sum of terms of the form

$$\sin m\phi \; P^m_{n-2r}(\cos\theta),$$

so that the theorem is proved for every function $f(\theta, \phi)$ which is of the prescribed form.

Alternative Method. These results can also be obtained as follows:—

A homogeneous rational integral function of degree n of the three variables $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$, when multiplied by r^n, becomes a homogeneous function $f_n(x, y, z)$ of x, y, and z. This function can be expressed in the form

$$f_n = Y_n + r^2 Y_{n-2} + r^4 Y_{n-4} + \ldots \qquad (38)$$

where Y_n is a solid spherical harmonic of degree n. To effect this, subtract from f_n the expression $(x^2 + y^2 + z^2)f_{n-2}$, where f_{n-2} is an arbitrary homogeneous function of x, y, z of degree $n - 2$, containing $\frac{1}{2}n(n - 1)$ arbitrary constants

Now if $V \equiv f_n - (x^2 + y^2 + z^2)f_{n-2}$ be substituted in $\nabla^2 V$, a function of degree $n - 2$ is obtained, and by equating the coefficients in this function to zero $\frac{1}{2}n(n - 1)$ equations are obtained, by means of which the arbitrary constants in f_{n-2} can be determined. V is then a spherical harmonic, say Y_n, so that

$$f_n = Y_n + r^2 f_{n-2}.$$

By applying this process to $f_{n-2}, f_{n-4}, \ldots$ in turn, till a function f_1 or f_0 is reached which, being of degree 1 or 0, is necessarily a spherical harmonic, we finally obtain the expression (38).

If now the equation (38) be divided by r^n, the resulting equation gives the expression for the given function in terms of surface spherical harmonics.

The evaluation of the harmonics Y_n, $Y_{n-2}, \ldots$ in (38) can be facilitated by means of the formula (p. 74)

$$\nabla^2(r^{2s} Y_{n-2s}) = 2s(2n - 2s + 1)r^{2s-2} Y_{n-2s}. \qquad (39)$$

By repeated applications of this formula to (38) we obtain the system of equations

$$
\begin{aligned}
\nabla^2 f_n &= 2(2n-1)Y_{n-2} + 4(2n-3)r^2Y_{n-4} + 6(2n-5)r^4Y_{n-6} + \ldots\\
(\nabla^2)^2 f_n &= 2\,.\,4(2n-3)(2n-5)Y_{n-4} + 4\,.\,6(2n-5)(2n-7)r^2Y_{n-6}\\
&\qquad + \ldots \qquad (40)\\
(\nabla^2)^3 f_n &= 2\,.\,4\,.\,6(2n-5)(2n-7)(2n-9)Y_{n-6} + \ldots\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

Finally we obtain the value of Y_0 or Y_1, according as n is even or odd; then from the previous equation we obtain Y_2 or Y_3, as the case may be, and so on

Example 1. Express

$f_5(x, y, z) = x^5 + y^5 + z^5 + 4x^3(y^2 + z^2) + 4y^3(z^2 + x^2) + 4z^3(x^2 + y^2)$ in the form (38).

$$
\begin{aligned}
\nabla^2 f_5 &= 36(x^3 + y^3 + z^3) + 24\Sigma x(y^2 + z^2)\\
(\nabla^2)^2 f_5 &= 312(x + y + z)
\end{aligned}
$$

But

$$
\begin{aligned}
f_5 &= Y_5 + r^2Y_3 + r^4Y_1,\\
\nabla^2 f_5 &= 18Y_3 + 28r^2Y_1,\\
(\nabla^2)^2 f_5 &= 280Y_1;
\end{aligned}
$$

and therefore

$$
\begin{aligned}
Y_1 &= \tfrac{39}{35}(x + y + z),\\
Y_3 &= \tfrac{2}{3}(x^3 + y^3 + z^3) - \tfrac{2}{5}r^2(x + y + z),\\
Y_5 &= -3(x^5 + y^5 + z^5) + \tfrac{10}{3}r^2(x^3 + y^3 + z^3) - \tfrac{5}{7}r^4(x + y + z)
\end{aligned}
$$

Example 2 If

$$\phi = yz^2 + zx^2 + xy^2 - y^2z - z^2x - x^2y + r^2(x + y + z),$$

show that

$$
\begin{aligned}
\phi r^3 = \tfrac{1}{6}(\sin\phi - \cos\phi)T_3^1(\mu) + \tfrac{1}{15}\cos 2\phi T_3^2(\mu) - \tfrac{1}{60}(\sin 3\phi + \cos 3\phi)T_3^3(\mu)\\
+ (\sin\phi + \cos\phi)\,T_1^1(\mu) + P_1(\mu)
\end{aligned}
$$

§ 13 **Expansion of an Arbitrary Function of θ and ϕ in a Series of Spherical Harmonics.** Let it be assumed that a function $f(\theta, \phi)$ can be expanded in a series

$$f(\theta, \phi) = \sum_{n=1}^{\infty}\left\{A_nP_n(\mu) + \sum_{m=1}^{n}(A_n^m \cos m\phi + B_n^m \sin m\phi)T_n^m(\mu)\right\} \qquad (41)$$

for $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq 2\pi$, and further that the series can be integrated term by term over these ranges Then, multiplying by $P_n(\mu)$, $\cos m\phi\, T_n^m(\mu)$, $\sin m\phi\, T_n^m(\mu)$ in turn, and integrating, we find that, from (18),

$$
\begin{aligned}
A_n &= \frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} f(\theta', \phi')P_n(\mu')d\mu\, d\phi',\\
A_n^m &= \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{1} f(\theta', \phi')T_n^m(\mu') \cos m\phi'\, d\mu' d\phi', \qquad (42)\\
B_n^m &= \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{1} f(\theta', \phi')T_n^m(\mu') \sin m\phi'\, d\mu' d\phi'.
\end{aligned}
$$

The validity of this expansion will not be investigated here; it holds usually for the functions which appear in connection with physical problems, and it is certainly true when the function $f(\theta, \phi)$ is continuous.

Example. Show that, if $0 \leq \theta \leq \pi$,

$$\sin\theta\cos 3\phi = 8\left\{7\frac{1}{6!}T_3^3(\mu) + 11\frac{2!}{8!}T_5^3(\mu) + 15\frac{4!}{10!}T_7^3(\mu) + \ldots\right\}\cos 3\phi.$$

Examples.

1. Show that, in the neighbourhood of the origin, every solution of Legendre's Associated Equation can be put in the form

$$A(x^2-1)^{\frac{1}{2}m}F\left(\frac{m-n}{2}, \frac{m+n+1}{2}, \tfrac{1}{2}, x^2\right)$$
$$+ B(x^2-1)^{\frac{1}{2}m}x\,F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, x^2\right).$$

2. Show that, if $k \neq l$,

$$\int_{-1}^{1} P_n^k(x)P_n^l(x)\frac{dx}{1-x^2} = 0.$$

3. Prove that

(i) $(n-m+1)P_{n+1}^m(x) - (2n+1)xP_n^m(x) + (n+m)P_{n-1}^m(x) = 0$;

(ii) $(n-m+1)Q_{n+1}^m(x) - (2n+1)xQ_n^m(x) + (n+m)Q_{n-1}^m(x) = 0$.

[For (i) differentiate (VI., 9) m times and (VI., 10) $m-1$ times and eliminate $\dfrac{d^{m-1}P_n(x)}{dx^{m-1}}$ between the resulting equations; similarly for (ii).]

4. If $\quad \cos\alpha = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)$

and $\quad \cos\beta = \cos\theta\cos\theta_2 + \sin\theta\sin\theta_2\cos(\phi-\phi_2)$,

show that

$$\int_0^{2\pi}\int_0^{\pi} P_n(\cos\alpha)P_n(\cos\beta)\sin\theta\,d\theta\,d\phi = \frac{4\pi}{2n+1}P_n(\cos\gamma),$$

where $\quad \cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$.

5. Show that:

(i) $a\cos^2\theta + b\sin^2\theta\cos^2\phi + c\sin^2\theta\sin^2\phi$

$$= \frac{a+b+c}{3} + \tfrac{2}{3}\left(a - \frac{b+c}{2}\right)P_2(\cos\theta) + \frac{b-c}{6}\cos 2\phi T_2^2(\cos\theta);$$

(ii) $\cos^2\theta\sin^2\theta\sin\phi\cos\phi = \frac{1}{105}\sin 2\phi T_4^2(\cos\theta)$
$+ \frac{1}{42}\sin 2\phi T_2^2(\cos\theta)$;

(iii) $\cos^3\theta\sin^3\theta\sin\phi\cos^2\phi = \sin 3\phi\{\frac{1}{6930}T_6^3(\cos\theta) + \frac{1}{1540}T_4^3(\cos\theta)\}$
$- \sin\phi\{\frac{2}{693}T_6^1(\cos\theta) - \frac{1}{770}T_4^1(\cos\theta) - \frac{1}{63}T_2^1(\cos\theta)\}$.

6. Show that

$$P_n(\cos\alpha\cos\beta) = P_n(\sin\alpha)P_n(0)$$
$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}T_n^m(\sin\alpha)T_n^m(0)\cos m\beta.$$

7. If m and n are positive integers ($m < n$), show that

$$T_n^m(x) = (-1)^m\frac{1}{2^n(n-m)!}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m}\frac{d^n}{dx^n}\{(x-1)^{n+m}(x+1)^{n-m}\}.$$

[Put $x + 1 = 2 + (x - 1)$ and use (5).]

8. Show that, if m is a positive integer,

$$\text{(i)}\quad \frac{d}{dx}P_n^m(x) = \frac{1}{\sqrt{(x^2-1)}}P_n^{m+1}(x) + \frac{mx}{x^2-1}P_n^m(x),$$

$$\text{(ii)}\quad \frac{d}{dx}Q_n^m(x) = \frac{1}{\sqrt{(x^2-1)}}Q_n^{m+1}(x) + \frac{mx}{x^2-1}Q_n^m(x).$$

9. If m and n are positive integers, and $m \leq n$, show that $Q_n^m(x)$ and $\frac{d}{dx}Q_n^m(x)$ do not vanish when x is zero.

[From *ex.* 11 of Ch. VI. it follows that $Q_n(0)$ and $Q'_n(0)$ cannot be zero, and therefore, since $Q^1_n(0) = \sqrt{(-1)}Q'_n(0)$, $Q'_n(0)$ is not zero. Hence, from the *example* at the end of § 1 of this chapter, $Q_n^m(0)$ cannot be zero for $m = 0, 1, 2, \ldots, n + 1$. Finally, from *ex.* 8 it follows that, for $m = 0, 1, 2, \ldots, n$, $\frac{d}{dx}Q_n^m(x)$ cannot be zero when $x = 0$.]

CHAPTER VIII

APPLICATIONS OF LEGENDRE COEFFICIENTS TO POTENTIAL THEORY

§ 1. **The Newtonian Law of Attraction.** In the theory of Gravitational Attraction it is assumed that any two particles of matter are attracted towards each other with a force which varies directly as the product of their masses and inversely as the square of the intervening distance; *i.e.*, if their masses are m_1 and m_2, and R is the distance between them, the force which each exerts on the other is equal to $k\frac{m_1 m_2}{R^2}$, where k is a gravitational constant, depending on the units of mass and distance employed: k is, in fact, the force of attraction between two unit masses at unit distance from each other In what follows it will be assumed that the units have been chosen so that the attraction between two unit masses at unit distance from each other is the unit of force, and thus k will have the value unity.

Force at a Point. If a particle of mass m be situated at a point $Q(\xi, \eta, \zeta)$ and a particle of unit mass at $P(x, y, z)$, the attraction exerted by the first particle on the second is equal to mR^{-2}, where

$$R^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2;$$

and the components X, Y, Z of this force are given by the equations

$$X = -m\frac{x - \xi}{R^3}, \quad Y = -m\frac{y - \eta}{R^3}, \quad Z = -m\frac{z - \zeta}{R^3}. \qquad (1)$$

It follows that the components of force at P due to a number of such particles m_1, m_2, . . . at the points Q_1, Q_2, . . . respectively are

$$X = -\sum_s m_s\frac{x - \xi_s}{R_s^3}, \quad Y = -\sum_s m_s\frac{y - \eta_s}{R_s^3}, \quad Z = -\sum_s m_s\frac{z - \zeta_s}{R_s^3}, \qquad (2)$$

where Q_s is the point (ξ_s, η_s, ζ_s) and $R_s = PQ_s$; while, for a continuous distribution of mass whose density at the point (ξ, η, ζ) is ρ,

$$\begin{aligned} X &= -\iiint \rho \frac{x - \xi}{R^3} d\xi d\eta d\zeta, \\ Y &= -\iiint \rho \frac{y - \eta}{R^3} d\xi d\eta d\zeta, \\ Z &= -\iiint \rho \frac{z - \zeta}{R^3} d\xi d\eta d\zeta, \end{aligned} \tag{3}$$

the triple integrals being taken over the volume of the attracting matter.

§ 2. **The Potential.** The formula (1) can be written

$$X = \frac{\partial}{\partial x}\left(\frac{m}{R}\right), \quad Y = \frac{\partial}{\partial y}\left(\frac{m}{R}\right), \quad Z = \frac{\partial}{\partial z}\left(\frac{m}{R}\right),$$

so that the components of the force at the point P due to the mass m at the point Q are the partial derivatives of the function m/R. This function is called the *Potential* at P of the mass m at Q, and is the work which would be done by the attraction due to the mass at Q (supposed fixed) in bringing unit mass from an infinite distance to the point P. If V is the potential at P due to a particle of mass m at Q, $V = m/R$; for a number of particles as in (2), $V = \sum_s m_s/R_s$; and for a continuous distribution as in (3)

$$V = \iiint \frac{\rho}{R} d\xi d\eta d\zeta. \quad . \quad . \quad . \quad (4)$$

In each case

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}, \quad . \quad . \quad (5)$$

and, for the component of force F along any path s,

$$F = \frac{\partial V}{\partial s}. \quad . \quad . \quad . \quad . \quad (6)$$

Note. For a distribution of attracting matter which does not extend to infinity, V tends to zero as P tends to infinity; and, indeed, rV tends to M, where r is the distance from any fixed point to P, and M is the total attracting matter. This follows from the formulæ for V given above, since r/R tends to unity as P tends to infinity.

Electric and Magnetic Potentials. In dealing with electric and magnetic potentials, it is found that the charges are of two kinds, described as positive and negative; and that like charges repel, unlike charges attract each other, according to the law m_1m_2/R^2. The potential in such cases is given by the same formula as before, and is the work done *against* the forces due to the system in bringing a positive unit from infinity to the point P. The component forces are then given by the equations

$$X = -\frac{\partial V}{\partial x}, \ Y = -\frac{\partial V}{\partial y}, \ Z = -\frac{\partial V}{\partial z}, \quad . \quad (5')$$

$$F = -\frac{\partial V}{\partial s}. \quad . \quad . \quad . \quad . \quad . \quad (6')$$

§ 3. **Some Properties of the Potential.** The following theorems on the potential due to masses which attract or repel according to the law m_1m_2/R^2 are of importance; proofs are to be found in all treatises on the mathematical theories of attraction and electrostatics.

Gauss's Theorem. Let S be a closed surface surrounding a region Σ containing attracting matter whose total mass is M; then if F is the normal component of the force at any point of S due to the attraction of the mass within S and also of any matter external to S,

$$\int F dS = -4\pi M, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the integral being taken over the whole surface S, and F being regarded as positive when the normal force acts away from Σ. If n denotes the normal measured away from Σ this equation may be written

$$\int \frac{\partial V}{\partial n} dS = -4\pi M. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For electrostatic forces these formulæ become

$$\int F dS = 4\pi M, \quad . \quad . \quad . \quad . \quad . \quad (7')$$

and

$$\int \frac{\partial V}{\partial n} dS = -4\pi M. \quad . \quad . \quad . \quad . \quad (8')$$

For a surface distribution of attracting matter of density σ this theorem leads to the formula

$$\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2} = -4\pi\sigma, \quad . \quad . \quad . \quad (9)$$

where n_1 and n_2 are the normals on each side of the surface measured away from the surface. (These normals are, of course, collinear.) For the surface distribution on an electric conductor it can be shown (cf. Ch. X., § 1) that the force at an internal point vanishes, and thus formula (9) becomes

$$\frac{\partial V}{\partial n} = -4\pi\sigma. \quad . \quad . \quad . \quad . \quad (9')$$

Gauss's Theorem can be established as follows. Consider an attracting particle of mass m at the point P, and let a cone of small solid angle $\delta\omega$ be generated by radii through P. This cone cuts the surface at the points Q_1, Q_2, . . . taken in order from P; the parts of the surface cut off by the cone at these points are δS_1, δS_2, . . ., and the outward-drawn normals are denoted by n_1, n_2, . . .

The attracting force at Q_1 is m/PQ_1^2, and the component of this along n_1 is $-m\cos\alpha\ PQ_1^{-2}$, where α is the angle between PQ_1 and n_1. Thus the outward normal force across δS_1 is $-m\,\delta S_1\cos\alpha/PQ_1^2$, and this is equal to $\mp m\,\delta\omega$, according as $\cos\alpha$ is positive or negative.

Now, if P lies outside Σ, the cone will cut S an even number of times, and the signs will be *plus* and *minus* alternately, so that the total normal force across δS_1, δS_2, . . . will be zero. On the other hand, if P lies inside Σ, the cone will cut S an odd number of times, the signs being *minus* and *plus* alternately, so that the total normal force across δS_1, δS_2, . . . will be $-m\delta\omega$. Hence the total normal force across S due to the particle at P is

$$-\int m\,d\omega = -4\pi m,$$

when P is inside, and is zero when P lies outside Σ. Thus, summing for all the particles within and outside Σ, we obtain Gauss's Theorem.

Laplace's Theorem. For points external to the attracting matter

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad . \quad . \quad (10)$$

This may be deduced from formula (4) by differentiation. The theorem also holds in the theories of electrostatics and magnetostatics.

Poisson's Theorem. For points within an attracting or repelling mass of density ρ

$$\nabla^2 V = -4\pi\rho. \quad . \quad . \quad . \quad (11)$$

To prove this theorem, put U = 1 in the formula (II., 6); then

$$\iiint \nabla^2 V d\Sigma = \iint \frac{\partial V}{\partial n} dS$$

$$= -4\pi \iiint \rho d\Sigma$$

by (8) and (8′). But this identity holds no matter how small the volume Σ may be; hence, at all points of the mass

$$\nabla^2 V = -4\pi\rho.$$

Laplace's Theorem is a particular case of Poisson's.

§ 4. **Expression for the Potential in terms of Spherical Harmonics.** For any distribution of attracting matter (or of electricity or magnetism) the potential V at a point $P(r, \theta, \phi)$ external to the matter can be expressed in the form

$$V = Y_0 + rY_1 + r^2 Y_2 + \ldots + r^n Y_n + \ldots$$

$$+ \frac{Z_0}{r} + \frac{Z_1}{r^2} + \frac{Z_2}{r^3} + \ldots + \frac{Z_n}{r^{n+1}} + \ldots, \quad . \quad (12)$$

where Y_n and Z_n are surface harmonics of degree n, provided that the sphere whose centre is the origin and radius r does not pass through any of the attracting matter.

For

$$V = \iiint \frac{\rho' r'^2 \sin\theta' dr' d\theta' d\phi'}{R},$$

where ρ' is the density at the point $Q(r', \theta', \phi')$ of the attracting matter and R = PQ, the integral being taken over the volume of the attracting matter. Now

$$R^2 = r^2 + r'^2 - 2rr' \cos\gamma,$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. Hence, if the volume of the attracting matter be divided into two parts

Σ and Σ', such that, throughout Σ, $r' > r$, while, throughout Σ', $r' < r$, then in the first region

$$\frac{1}{R} = \frac{1}{r'}P_0(\cos\gamma) + \frac{r}{r'^2}P_1(\cos\gamma) + \ldots + \frac{r^n}{r'^{n+1}}P_n(\cos\gamma) + \ldots,$$

and in the second region

$$\frac{1}{R} = \frac{1}{r}P_0(\cos\gamma) + \frac{r'}{r^2}P_1(\cos\gamma) + \ldots + \frac{r'^n}{r^{n+1}}P_n(\cos\gamma) + \ldots.$$

Accordingly

$$V = \sum_{n=0}^{\infty} r^n \iiint \frac{\rho'}{r'^{n-1}} P_n(\cos\gamma) \sin\theta' \, dr' d\theta' d\phi'$$

$$+ \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \iiint \rho' r'^{n+2} P_n(\cos\gamma) \sin\theta' \, dr' d\theta' d\phi',$$

where the integrals in the first summation are taken over the volume Σ, and those in the second over the volume Σ'.

If in these integrals we expand $P_n(\cos\gamma)$ by means of the Addition Theorem (VII, 34), we obtain an expression of the type (12), where Y_n and Z_n are of the form

$$A_0 P_n(\cos\theta) + \sum_{m=1}^{n} (A_m \cos m\phi + B_m \sin m\phi) T_n^m(\cos\theta),$$

and the A's and B's are constants.

Note. Since $P_0(\cos\gamma) = 1$,

$$Z_0 = \iiint \rho' r'^2 \sin\theta \, dr' d\theta' d\phi' = M,$$

where M is the mass of attracting matter which lies in the region Σ'.

Corollary 1. If all the attracting matter lies within the sphere with the origin as centre and r as radius,

$$V = \frac{Z_0}{r} + \frac{Z_1}{r^2} + \frac{Z_2}{r^3} + \ldots.$$

Corollary 2. If all the attracting matter lies outside the sphere with the origin as centre and r as radius,

$$V = Y_0 + rY_1 + r^2 Y_2 + \ldots$$

Corollary 3. If the distribution of attracting matter is symmetrical about the z-axis,

$$V = \left(A_0 + \frac{B_0}{r}\right)P_0(\cos\theta) + \left(A_1 r + \frac{B_1}{r^2}\right)P_1(\cos\theta)$$
$$+ \left(A_2 r^2 + \frac{B_2}{r^3}\right)P_2(\cos\theta) + \dots \quad (13)$$

For the coefficient of $\frac{\sin}{\cos}(m\phi)T_n^m(\cos\theta)$ in Y_n or Z_n is of the form

$$\iiint \rho' r'^k \frac{\sin}{\cos}(m\phi')T_n^m(\cos\theta') \sin\theta' dr' d\theta' d\phi',$$

and, since ρ' is independent of ϕ', this contains a factor

$$\int_0^{2\pi} \frac{\sin}{\cos}(m\phi')\, d\phi',$$

which vanishes except when m is zero.

Corollary 4. The function $l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z}$, where l, m, n are constants, can be expressed in the form

$$Y_0 + rY_1 + r^2Y_2 + r^3Y_3 + \dots$$
$$+ \frac{Z_1}{r^2} + \frac{Z_2}{r^3} + \frac{Z_3}{r^4} + \dots$$

For, let V_p be a solid spherical harmonic of integral degree p, and let $U_p = l\frac{\partial V_p}{\partial x} + m\frac{\partial V_p}{\partial y} + n\frac{\partial V_p}{\partial z}$. Then, if p is a positive integer, U_p is a homogeneous polynomial in x, y, z of degree $p - 1$. But

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\nabla^2 V_p = 0$$

and therefore

$$\nabla^2\left(l\frac{\partial V_p}{\partial x} + m\frac{\partial V_p}{\partial y} + n\frac{\partial V_p}{\partial z}\right) = 0,$$

so that U_p is a solid spherical harmonic of degree $p - 1$. If p is zero, $U_p = 0$. Again, if p is a negative integer $-q-1$,

$$V_{-q-1} = \frac{W_q}{r^{2q+1}},$$

where W_q is a solid spherical harmonic of degree q; thus

$$U_p = \frac{K_{q+1}}{r^{2q+3}},$$

where K_{q+1} is a homogeneous polynomial of degree $q + 1$. But this function satisfies Laplace's Equation; hence K_{q+1} is a solid spherical harmonic of degree $q + 1$, and

$$U_p = \frac{S_{q+1}}{r^{q+2}},$$

where S_{q+1} is a surface harmonic of degree $q + 1$. Thus, if the operator $l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}$ be applied to (12), we obtain the required expression for $l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z}$.

§ 5 **Potential of a Thin Uniform Wire bent into the form of a Circle.** Let c be the radius of the circle, and M the mass of the wire Take the centre of the circle as the origin O, and the perpendicular through O to the plane of the circle as the z-axis. The potential of the wire at the point (o, o, z) is then $\frac{M}{\sqrt{(c^2 + z^2)}}$, which, for $|z| < c$, can be expanded in the form

$$\frac{M}{c}\left\{1 - \frac{1}{2}\frac{z^2}{c^2} + \frac{1 \cdot 3}{2 \cdot 4}\frac{z^4}{c^4} - \quad . \right\} \quad . \quad . \quad (14)$$

But, by Corollaries 2 and 3 of § 4, the potential V at any point $P(r, \theta, \phi)$ for which $r < c$ can be expressed in the form

$$V = A_0P_0(\cos\theta) + A_1rP_1(\cos\theta) + A_2r^2P_2(\cos\theta) + \ . \ . \ . \quad (15)$$

Now, at points on the positive z-axis, $z = r$, $\theta = 0$, and $P_n(1) = 1$, so that (15) becomes

$$V = A_0 + A_1z + A_2z^2 + \ . \ . \ . \quad . \quad . \quad (16)$$

But, for all values of z such that $-c < z < c$ this must be equal to (14), and thus the coefficients of the various powers of z in (14) and (16) must be equal. Hence (15) can be written

$$V = \frac{M}{c}\left\{P_0(\cos\theta) - \frac{1}{2}\frac{r^2}{c^2}P_2(\cos\theta) + \frac{1 \cdot 3}{2 \cdot 4}\frac{r^4}{c^4}P_4(\cos\theta) - \ . \ . \ . \right\}. \quad (17)$$

Similarly, when $r > c$,

$$\frac{M}{\sqrt{(c^2+z^2)}} = \frac{M}{z}\left\{1 - \frac{1}{2}\frac{c^2}{z^2} + \frac{1 \cdot 3}{2 \cdot 4}\frac{c^4}{z^4} - \ldots\right\},$$

and $V = A_0\frac{1}{r}P_0(\cos\theta) + A_1\frac{1}{r^2}P_1(\cos\theta) + A_2\frac{1}{r^3}P_2(\cos\theta) + \ldots$

so that

$$V = \frac{M}{r}\left\{P_0(\cos\theta) - \frac{1}{2}\frac{c^2}{r^2}P_2(\cos\theta) + \frac{1 \cdot 3}{2 \cdot 4}\frac{c^4}{r^4}P_4(\cos\theta) - \ldots\right\} \tag{18}$$

If we apply the substitution (V., 46)

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\pi\left\{\frac{z + \sqrt{(z^2 - r^2)}\cos\psi}{r}\right\}^n d\psi$$

to the terms of (17), we obtain the formula

$$V = \frac{M}{\pi}\int_0^\pi \frac{d\psi}{[c^2 + \{z + \sqrt{(z^2 - r^2)}\cos\psi\}^2]^{\frac{1}{2}}};$$

and similarly, from the formula (V., 48)

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\pi\left\{\frac{z + \sqrt{(z^2 - r^2)}\cos\psi}{r}\right\}^{-n-1} d\psi$$

and (18) we obtain the same result.

Example. If M is the mass of a uniform thin hemispherical shell of radius c and if the centre is taken as the origin with the shell entirely above the (x, y) plane, prove that its potential V is given by

$$V = \frac{M}{c}\left[P_0(\cos\theta) + \frac{1}{2}\left(\frac{r}{c}\right)P_1(\cos\theta) - \frac{1 \cdot 1}{2 \cdot 4}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\left(\frac{r}{c}\right)^5 P_5(\cos\theta) - \ldots\right], \text{ if } r < c,$$

$$V = \frac{M}{c}\left[\frac{c}{r}P_0(\cos\theta) + \frac{1}{2}\left(\frac{c}{r}\right)^2 P_1(\cos\theta) - \frac{1 \cdot 1}{2 \cdot 4}\left(\frac{c}{r}\right)^4 P_3(\cos\theta) + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\left(\frac{c}{r}\right)^6 P_5(\cos\theta) - \ldots\right], \text{ if } r > c.$$

§ 6. Attraction of a Uniform Circular Lamina.

Theorem. The solid angle subtended by a closed plane curve at any point P, is proportional to the component attraction at the point perpendicular to the plane of the curve, due to a plane lamina of uniform density and thickness, bounded by the closed plane curve.

For the solid angle Ω is given by

$$\Omega = \iint \frac{\cos\theta}{r^2} dS,$$

where r is the distance from P to a point Q on the lamina, θ is the angle between PQ and PR, the perpendicular from P to the plane of the lamina, and the integral is taken over the area of the lamina. Also, if ρ is the density, and k the thickness of the lamina, the component attraction in the direction PR is

$$\iint \frac{k\rho}{r^2} \cos\theta \, dS = k\rho\Omega,$$

and thus the theorem has been proved.

Expression for the Solid Angle subtended at any point by a Circle in terms of Legendre Coefficients. Take the origin at the centre of the circle, and the z-axis perpendicular to the plane of the circle; then if V is the potential of a uniform plane lamina bounded by the circle, the component force due to the attraction at any point parallel to the z-axis will be $\frac{\partial V}{\partial z}$ or $-k\rho\Omega$, where Ω is the solid angle subtended by the circle at the point. Hence, assuming that the formula (13) holds for V, we deduce from § 4, Corollary 4 that Ω can be expressed in the form

$$\Omega = A_0 P_0(\cos\theta) + \left(A_1 r + \frac{B_1}{r^2}\right) P_1(\cos\theta)$$
$$+ \left(A_2 r^2 + \frac{B_2}{r^3}\right) P_2(\cos\theta) + \dots .$$

But if Q is the point (o, o, z) on the positive z-axis, and E is any point on the circle, the value Ω_Q of Ω at Q, being the solid angle sustended at Q by the part of the sphere of centre Q and radius QE bounded by the circle, is

$$\Omega_Q = \int_0^\theta 2\pi r \sin\theta' \, r d\theta' / r^2,$$

where $r =$ QE and θ is the angle OQE: thus, if c is the radius of the circle,

$$\Omega_Q = 2\pi(1 - \cos\theta) = 2\pi\left\{1 - \frac{z}{\sqrt{(c^2 + z^2)}}\right\}$$
$$= 2\pi\left\{1 - \frac{z}{c} + \frac{1}{2}\left(\frac{z}{c}\right)^3 - \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{z}{c}\right)^5 + \dots\right\}, \text{ if } z < c$$
$$= 2\pi\left\{\frac{1}{2}\left(\frac{c}{z}\right)^2 - \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{c}{z}\right)^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{c}{z}\right)^6 - \dots\right\}, \text{ if } z > c.$$

Hence, as in the previous section, for any point for which $z > 0$

$$\Omega = 2\pi\left\{P_0(\cos\theta) - \frac{r}{c}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) - \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{c}\right)^5 P_5(\cos\theta) + \ldots\right\}, \text{ if } r < c \tag{19}$$

$$= 2\pi\left\{\frac{1}{2}\left(\frac{c}{r}\right)^2 P_1(\cos\theta) - \frac{1\cdot 3}{2\cdot 4}\left(\frac{c}{r}\right)^4 P_3(\cos\theta) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{c}{r}\right)^6 P_5(\cos\theta) - \ldots\right\}, \text{ if } r > c.$$

For points at which $z < 0$ these formulae become

$$\Omega = 2\pi\left\{P_0(\cos\theta) + \frac{r}{c}P_1(\cos\theta) - \frac{1}{2}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{c}\right)^5 P_5(\cos\theta) - \ldots\right\}, \text{ if } r < c \tag{19'}$$

$$= 2\pi\left\{-\frac{1}{2}\left(\frac{c}{r}\right)^2 P_1(\cos\theta) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{c}{r}\right)^4 P_3(\cos\theta) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{c}{r}\right)^6 P_5(\cos\theta) + \ldots\right\}, \text{ if } r > c.$$

Corollary. $\Omega = 2\pi - 2\int_0^\pi \frac{|z| + \sqrt{(z^2 - r^2)}\cos\psi}{[c^2 + \{|z| + \sqrt{(z^2 - r^2)}\cos\psi\}^2]^{\frac{3}{2}}}\,d\psi.$

Potential of a Uniform Circular Lamina It has been shown above that if V is the required potential, then at any point on the positive z-axis

$$-\frac{\partial V}{\partial z} = k\rho 2\pi\left\{1 - \frac{z}{\sqrt{(c^2 + z^2)}}\right\},$$

so that

$$V = 2\pi k\rho\{\sqrt{(c^2 + z^2)} - z\} + C.$$

Now, when z tends to infinity, the expression in the bracket tends to zero, as can be seen by expanding it in descending powers of z; but V also tends to zero, so that C must vanish. Hence

$$V = \frac{2M}{c^2}\{\sqrt{(c^2 + z^2)} - z\},$$

where M is the mass of the lamina. Thus

$$V = \frac{2M}{c^2}\left\{c - z + \frac{1}{2}\frac{z^2}{c} - \frac{1\,.\,1}{2\,.\,4}\frac{z^4}{c^3} + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\frac{z^6}{c^5} - \;.\;.\;.\right\}, \text{ if } z < c$$

$$= \frac{2M}{c^2}\left\{\frac{1}{2}\frac{c^2}{z} - \frac{1\,.\,1}{2\,.\,4}\frac{c^4}{z^3} + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\frac{c^6}{z^5} - \;.\;.\;.\right\}, \text{ if } z > c.$$

The potential at any point for which $z > 0$ is therefore given by

$$V = \frac{2M}{c}\left\{P_0(\cos\theta) - \frac{r}{c}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{c}\right)^2 P_2(\cos\theta)\right.$$

$$\left. - \frac{1\,.\,1}{2\,.\,4}\left(\frac{r}{c}\right)^4 P_4(\cos\theta) + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{r}{c}\right)^6 P_6(\cos\theta) - \;.\;.\;.\right\}, \text{ if } r < c$$

$$(20)$$

$$= \frac{2M}{c}\left\{\frac{1}{2}\frac{c}{r}P_0(\cos\theta) - \frac{1\,.\,1}{2\,.\,4}\left(\frac{c}{r}\right)^3 P_2(\cos\theta)\right.$$

$$\left. + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{c}{r}\right)^5 P_4(\cos\theta) - \;.\;.\;.\right\}, \text{ if } r > c.$$

For $z < 0$ the sign of the second term in the first formula is changed, while all the other terms remain unaltered.

Corollary.

$$V = \frac{2M}{\pi c^2}\int_0^\pi [c^2 + \{|z| + \sqrt{(z^2 - r^2)}\cos\psi\}^2]^{\frac{1}{2}}d\psi - \frac{2M|z|}{c^2}.$$

§ 7. **Potentials of a Magnet and of a Magnetic Shell.** Let A and B be the positive and negative poles of a simple magnet, m and $-m$ the strengths of these poles. Then BA is the *Axis*, C, the mid-point of BA, is the *Centre*, and $2am$ or M, where BA $= 2a$, is the *Magnetic Moment* of the magnet. The direction BA is the *Direction of Magnetisation*.

Potential of a Simple Magnet. Let r be the distance of a point P from C, and let θ be the angle ACP; then the potential V at P is given by

$$V = \frac{m}{AP} - \frac{m}{BP} = \frac{m}{\sqrt{(r^2 - 2ar\cos\theta + a^2)}} - \frac{m}{\sqrt{(r^2 + 2ar\cos\theta + a^2)}}$$

$$= M\left\{\frac{1}{r^2}P_1(\cos\theta) + \frac{a^2}{r^4}P_3(\cos\theta) + \;.\;.\;.\right\}, \quad (21)$$

provided that $r > a$.

For a *small* magnet, when $(a/r)^3$ is so small that it may be neglected, the potential of the magnet is taken to be

$$V = \frac{M}{r^2}P_1(\cos\theta) = \frac{M\cos\theta}{r^2}. \qquad . \qquad . \qquad (22)$$

Magnetic Shells. A magnetic shell is a thin sheet of magnetisable substance, magnetised at each point in the direction of the normal to the sheet at that point. If δS is a small area of the shell at a point Q, and δM the magnetic moment of this portion of the shell, the *Strength* of the shell at the point Q is $\delta M/\delta S$. If the strength is the same at every point, the shell is said to be *Uniform.*

From (22) it follows that the potential at a point P of a thin uniform magnetic shell of strength J is

$$\int\frac{J\cos\theta}{r^2}dS,$$

where $r = QP$, θ is the angle between QP and the direction of magnetisation at Q, and the integral is taken over the area of the shell. Thus the potential at P is $J\Omega$, where Ω is the solid angle subtended by the shell at P.

If now the shell is bounded by the circle $z = 0$, $x^2 + y^2 = c^2$, and has its positive face on the side towards the positive z-axis, it follows from (19) that the potential for a point on the positive side of the shell is

$$\begin{aligned} V &= 2\pi J\left\{P_0(\cos\theta) - \frac{r}{c}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{c}\right)^3 P_3(\cos\theta)\right. \\ &\qquad\left. - \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{c}\right)^5 P_5(\cos\theta) + \ldots\right\}, \text{ if } r < c, \\ &= 2\pi J\left\{\frac{1}{2}\left(\frac{c}{r}\right)^2 P_1(\cos\theta) - \frac{1\cdot 3}{2\cdot 4}\left(\frac{c}{r}\right)^4 P_3(\cos\theta)\right. \\ &\qquad\left. + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{c}{r}\right)^6 P_5(\cos\theta) - \ldots\right\}, \text{ if } r > c. \end{aligned}$$

CHAPTER IX

POTENTIALS OF SPHERICAL SHELLS, SPHERES, AND SPHEROIDS

§ 1. **General Theorems on the Potential.** The following theorems will be found useful in some of the further applications of the Spherical Harmonics to Potential Theory.

Theorem 1. If the functions V_1 and V_2 satisfy Laplace's Equation throughout a region Σ, and are equal at all points of S, the bounding surface of Σ, V_1 and V_2 are equal at all points of Σ.

For, let $V_1 - V_2 = V$; then the function V has the value zero at all points of S, and satisfies Laplace's Equation throughout Σ. Hence, if in Green's Theorem (II., 6) we put V for U, we get

$$\iiint\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\}d\Sigma = 0.$$

Here the integrand cannot be negative, and therefore the equation can only be satisfied if

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

vanishes at all points of Σ; this, again, can only be the case if each term is zero, and so, throughout Σ,

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial z} = 0.$$

Hence V is constant, and, as its value on S is zero, it must vanish throughout Σ. Thus, at all points of Σ, the functions V_1 and V_2 are equal.

Theorem 2. If the functions V_1 and V_2 satisfy Laplace's Equation throughout the region Σ, and if $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ at all points of the surface S, the difference between V_1 and V_2 remains constant throughout Σ.

For, if $V = V_1 - V_2$, $\frac{\partial V}{\partial n}$ vanishes at all points of S, and V satisfies Laplace's Equation throughout Σ. Now in Green's Theorem (II., 6) put V for U, and as in *Theorem* 1 we get

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial z} = 0$$

throughout Σ, so that V and therefore $V_1 - V_2$ is constant.

Note. If the region extends to infinity, V must vanish there, and therefore V_1 and V_2 are equal. If the region does not extend to infinity, $V_1 = V_2 + C$, where C is a constant. The presence of the constant is due to the fact that a constant potential gives rise to no forces

§ 2. **Potential of a Thin Spherical Shell of Given Surface Density.** Consider a spherical shell whose thickness is so small that it may be neglected, and let the origin be the centre, a the radius, and σ or $\sigma(\theta', \phi')$ the surface density of the matter composing the shell at the point (a, θ', ϕ') Then the potential of the shell at the point $P(r, \theta, \phi)$ is

$$V = \int_0^{2\pi}\int_{-1}^{1} \frac{\sigma a^2 d\mu' d\phi'}{R}, \quad . \qquad (1)$$

where $\qquad R^2 = r^2 + a^2 - 2ra \cos \gamma, \qquad . \quad (2)$

and $\qquad \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \qquad (3)$

If P lies outside the sphere, $r > a$, and $1/R$ can be expanded in descending powers of r, so that the potential at an external point is

$$V_e = \int_0^{2\pi}\int_{-1}^{1} \sigma a^2 d\mu' d\phi' \left\{ \frac{1}{r} P_0(\cos \gamma) + \frac{a}{r^2} P_1(\cos \gamma) + \frac{a^2}{r^3} P_2(\cos \gamma) + \ . \ . \ . \right\}$$

$$= \sum_{n=0}^{\infty} \frac{a^{n+2}}{r^{n+1}} \int_0^{2\pi}\int_{-1}^{1} \sigma P_n(\cos \gamma) d\mu' d\phi'.$$

As in Ch. VIII, § 4, it follows that

$$V_e = \sum_{n=0}^{\infty} \frac{a^{n+2}}{r^{n+1}} Y_n(\theta, \phi), \quad . \quad . \quad . \qquad (4)$$

where
$$Y_n(\theta, \phi) = \int_0^{2\pi}\int_{-1}^{1} \sigma P_n(\cos\gamma)\, d\mu'\, d\phi' \quad . \quad . \quad (5)$$

is a surface harmonic of degree n.

Similarly, if P is an internal point, $r < a$, and the potential is

$$V_i = \int_0^{2\pi}\int_{-1}^{1} \sigma a^2 d\mu' d\phi' \left\{ \frac{1}{a} P_0(\cos\gamma) + \frac{r}{a^2} P_1(\cos\gamma) + \frac{r^2}{a^3} P_2(\cos\gamma) + \dots \right\}$$

$$= \sum_{n=0}^{\infty} \frac{r^n}{a^{n-1}} Y_n(\theta, \phi). \quad . \quad . \quad . \quad (6)$$

Note 1. The first term in (4) is M/r, where M is the mass of the shell; for

$$M = \int_0^{2\pi}\int_{-1}^{1} \sigma a^2 d\mu' d\phi'$$

Similarly the first term in (6) is M/a. [Cf. Ch. VIII., § 4, *Note.*]

Again, assume that, as in VII (41), (42), the function $\sigma(\theta', \phi')$ can be expanded in a series

$$\sigma(\theta', \phi') = \sum_{m=0}^{\infty} Z_m(\theta', \phi'), \quad . \quad . \quad . \quad (7)$$

where $Z_m(\theta', \phi')$ is a surface spherical harmonic of integral degree m, and also that the series obtained by substituting (7) in (5) can be integrated term by term. Then, from VII. (29) and VII. (33) we deduce that

$$Y_n(\theta, \phi) = \frac{4\pi}{2n + 1} Z_n(\theta, \phi),$$

and, consequently, that

$$V_e = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n + 1} \frac{a^{n+2}}{r^{n+1}} Z_n(\theta, \phi), \quad . \quad . \quad (8)$$

and
$$V_i = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n + 1} \frac{r^n}{a^{n-1}} Z_n(\theta, \phi). \quad . \quad . \quad (9)$$

Note 2. If the series (8) and (9) are convergent when $r = a$, the potential at a point on the sphere is

$$V = 4\pi a \sum_{n=0}^{\infty} \frac{1}{2n+1} Z_n(\theta, \phi). \quad . \quad . \quad (10)$$

Note 3. If $\sigma = Z_n(\theta', \phi')$,

$$V_e = \frac{4\pi}{2n+1} \frac{a^{n+2}}{r^{n+1}} Z_n(\theta, \phi), \ V_i = \frac{4\pi}{2n+1} \frac{r^n}{a^{n-1}} Z_n(\theta, \phi).$$

In particular, if σ is constant, (8) and (9) reduce to the well-known results

$$V_e = \frac{M}{r}, \ V_i = \frac{M}{a}.$$

Example 1. If $\sigma = c \cos\theta'$, where c is a constant, show that

$$V_e = \frac{4\pi}{3} c \frac{a^3}{r^2} \cos\theta, \ V_i = \frac{4\pi}{3} c r \cos\theta.$$

Example 2. If $\sigma = A \cos^2\theta' + B \sin^2\theta' \cos^2\phi' + C \sin^2\theta' \sin^2\phi'$, show that

$$V_e = 4\pi \frac{a^2}{r} Y_0(\theta, \phi) + \frac{4\pi}{5} \frac{a^4}{r^3} Y_2(\theta, \phi),$$

$$V_i = 4\pi a Y_0(\theta, \phi) + \frac{4\pi}{5} \frac{r^2}{a} Y_2(\theta, \phi),$$

where $$Y_0(\theta, \phi) = \tfrac{1}{3}(A + B + C)$$
and
$$Y_2(\theta, \phi) = \tfrac{1}{3}(2A - B - C)P_2(\cos\theta) + \tfrac{1}{6}(B - C) \cos 2\phi T_2^2(\cos\theta).$$
[Cf. Ch. VII., *ex*. 5]

Example 3. The surface density at any point (x', y', z') of a spherical shell with the origin as centre and a as radius is $\sigma = mxy$; show that

$$V_i = \frac{4\pi am}{5} xy, \quad V_e = \frac{4\pi am}{5} xy \left(\frac{a}{r}\right)^5.$$

Example 4. Matter of mass M is distributed on a spherical surface $r = a$ so that its density at any point is proportional to the square of the distance of the point from the point (o, o, b) outside the sphere; prove that

$$V_e = M\left\{\frac{1}{r} - \frac{2a^2 b}{3(a^2 + b^2)} \frac{z}{r^3}\right\}.$$

§ 3. Potential of a Thin Spherical Shell when the Value of the Potential on the Surface is given. First of all, suppose that the potential V on the surface is equal to $Y_n(\theta, \phi)$, where

$Y_n(\theta, \phi)$ is a surface harmonic of integral degree n; then the function $V_i = \left(\frac{r}{a}\right)^n Y_n(\theta, \phi)$ satisfies Laplace's Equation at all points within the sphere, and is equal to V on the surface of the sphere. From Theorem 1 of § 1 it follows that V_i is the value of the potential within the sphere. Similarly, since $V_e = \left(\frac{a}{r}\right)^{n+1} Y_n(\theta, \phi)$ is equal to V on the surface of the sphere, and, like V, tends to zero when r tends to infinity, V_e is the value of the potential outside the sphere.

Next, suppose that, as in VII. (41), V has been expressed in the form

$$V = \sum_{n=0}^{\infty} Y_n(\theta, \phi); \quad . \quad . \quad . \quad (11)$$

then it follows as in the particular case just considered that

$$V_i = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n Y_n(\theta, \phi), \quad . \quad . \quad . \quad (12)$$

and

$$V_e = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} Y_n(\theta, \phi). \quad . \quad . \quad (13)$$

From the formula (VIII, 9)

$$\sigma = \frac{1}{4\pi} \operatorname*{Lim}_{r \to a} \left\{ -\frac{\partial V_e}{\partial r} + \frac{\partial V_i}{\partial r} \right\}$$

we deduce that

$$\sigma = \frac{1}{4\pi a} \sum_{n=0}^{\infty} (2n+1) Y_n(\theta, \phi). \quad . \quad . \quad (14)$$

Note. By putting $\frac{2n+1}{4\pi a} Y_n(\theta, \phi)$ for $Z_n(\theta, \phi)$ in (10) we can deduce (12), (13), and (14) from (9), (8), and (7) respectively.

§ 4. **A Theorem on Inverse Points.** If P_0 and P_1 are inverse points with regard to the sphere $r = a$ (*i.e.*, if O, P_0, and P_1 are collinear, and $r_0 r_1 = a^2$, where $OP_0 = r_0$ and $OP_1 = r_1$), the potentials V_0 and V_1 of the surface density of the sphere at these points are connected by the relation

$$\frac{V_0}{V_1} = \frac{a}{r_0} = \frac{r_1}{a}. \qquad . \quad . \quad . \quad (15)$$

Suppose, for example, that P_0 is an internal and P_1 an external point; then, by (12),

$$V_0 = \sum_{n=0}^{\infty} \left(\frac{r_0}{a}\right)^n Y_n(\theta, \phi)$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{r_1}\right)^n Y_n(\theta, \phi)$$

$$= \frac{r_1}{a} V_1,$$

by (13).

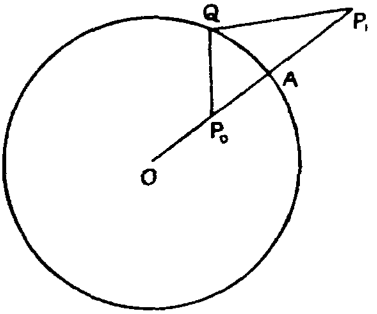

FIG. 17.

Note 1. This theorem can also be deduced from the geometry of inverse points. For, let a particle of mass m be situated at a point Q (Fig. 17) on the sphere, and let v_0 and v_1 be the potentials of this particle at P_0 and P_1. If OP_0P_1 cuts the sphere in A, QA bisects the angle P_0QP_1. Then

$$\frac{v_0}{v_1} = \frac{m/QP_0}{m/QP_1} = \frac{QP_1}{QP_0} = \frac{AP_1}{P_0A} = \frac{r_1 - a}{a - r_0} = \frac{r_1 - a}{a - a^2/r_1} = \frac{r_1}{a};$$

and, as this holds for all positions of Q on the sphere, it follows, by addition, that

$$\frac{V_0}{V_1} = \frac{r_1}{a}.$$

Note 2. If particles of masses m_0 and m_1, where

$$\frac{m_0}{m_1} = \frac{r_0}{a} = \frac{a}{r_1},$$

are situated at P_0 and P_1, their potentials V_0 and V_1 at any point of the sphere are equal: for

$$\frac{V_0}{V_1} = \frac{m_0/P_0Q}{m_1/P_1Q} = 1.$$

§ 5. **Poisson's Integrals.** If $V(\theta', \phi')$, the value of V on the sphere $r = a$, is given, it is possible to express V_i and V_e in a form in which the spherical harmonics do not appear.

For, from (11) and (VII., 41, 42),

$$\begin{aligned} Y_n(\theta, \phi) &= \frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} V(\theta', \phi')\Big\{P_n(\mu)P_n(\mu') \\ &\qquad + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\cos m(\phi - \phi')T_n^m(\mu)T_n^m(\mu')\Big\}d\mu' d\phi' \\ &= \frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} V(\theta', \phi')P_n(\cos\gamma)d\mu' d\phi', \end{aligned}$$

by (VII., 34); and therefore, from (12),

$$\begin{aligned} V_i &= \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n\int_0^{2\pi}\int_{-1}^{1} V(\theta', \phi')P_n(\cos\gamma)d\mu' d\phi' \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} V(\theta', \phi')\left[\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\gamma)\right]d\mu' d\phi'. \end{aligned}$$

But, by (V., 38), the sum of the series in the square bracket is

$$\frac{a(a^2 - r^2)}{(a^2 - 2ar\cos\gamma + r^2)^{\frac{3}{2}}}.$$

Hence, if $r < a$,

$$V_i = \frac{a(a^2 - r^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{V(\theta', \phi')d\mu' d\phi'}{(r^2 - 2ar\cos\gamma + a^2)^{\frac{3}{2}}}. \qquad (16)$$

Similarly it can be shown that, for $r > a$,

$$V_e = \frac{a(r^2 - a^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{V(\theta', \phi')d\mu' d\phi'}{(r^2 - 2ar\cos\gamma + a^2)^{\frac{3}{2}}}. \qquad (17)$$

§ 6. **Potential of a Thick Spherical Shell when the Density is given.** Next, consider a shell of attracting matter and the coefficients A, A_m, B_m are functions of r' alone. As-bounded by the concentric spheres $r = a$, $r = b$, where $a < b$, and let ρ or $\rho(r', \theta', \phi')$ be the density at the point (r', θ', ϕ') of the shell; then the potential at the point $P(r, \theta, \phi)$ is

$$V = \int_a^b \int_0^{2\pi} \int_{-1}^{1} \frac{\rho(r', \theta', \phi') r'^2 d\mu' d\phi' dr'}{R}, \quad . \quad (18)$$

where $\quad R^2 = r^2 + r'^2 - 2rr' \cos \gamma,$

and $\quad \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$

For points external to the sphere $r > r'$, and $1/R$ can be expanded in descending powers of r; proceeding as in § 2 we find that the potential is

$$V_e = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} Y_n(\theta, \phi), \quad . \quad . \quad . \quad (19)$$

where

$$Y_n(\theta, \phi) = \int_a^b \int_0^{2\pi} \int_{-1}^{1} \rho(r', \theta', \phi') r'^{n+2} P_n(\cos \gamma) d\mu' d\phi' dr' \quad (20)$$

is a surface harmonic of degree n.

Similarly, at points in the spherical hollow within the shell $r < r'$, and

$$V_i = \sum_{n=0}^{\infty} r^n Z_n(\theta, \phi), \quad . \quad . \quad . \quad (21)$$

where

$$Z_n(\theta, \phi) = \int_a^b \int_0^{2\pi} \int_{-1}^{1} \rho(r', \theta', \phi') r'^{-n+1} P_n(\cos \gamma) d\mu' d\phi' dr'. \quad (22)$$

Note 1. The series (19) holds when $r = b$, and the series (21) when $r = a$.

Note 2. The first term in (19) is M/r, where M is the mass of the shell.

Now, if ρ be regarded, for any value of r', as a function of θ' and ϕ', it may be expanded, by (VII., 41), in the form

$$\rho(r', \theta', \phi') = \sum_{n=0}^{\infty} X_n(r', \theta', \phi') \quad . \quad . \quad (23)$$

where

$$X_n(r', \theta', \phi') = AP_n(\mu') + \sum_{m=1}^{n} \{A_m \cos m\phi' + B_m \sin m\phi'\} T_n^m(\mu'), \quad (24)$$

suming that the series (23) can be integrated term by term over the volume of the shell, we deduce from (20) and (23) that

$$Y_n(\theta, \phi) = \int_a^b \int_0^{2\pi} \int_{-1}^{1} X_n(r', \theta', \phi') r'^{n+2} P_n(\cos\gamma) d\mu' d\phi' dr'$$

$$= \frac{4\pi}{2n+1}\left\{CP_n(\mu) + \sum_{m=1}^{n} (C_m \cos m\phi + D_m \sin m\phi) T_n^m(\mu)\right\}, \qquad (25)$$

by (VII, 33), where $C = \int_a^b A r'^{n+2} dr'$,

$$C_m = \int_a^b A_m r'^{n+2} dr', \qquad D_m = \int_a^b B_m r'^{n+2} dr'. \qquad (26)$$

Thus

$$V_e = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{1}{r^{n+1}} \int_a^b X_n(r', \theta, \phi) r'^{n+2} dr'. \qquad (27)$$

Similarly, from (22) and (23)

$$Z_n(\theta, \phi) = \int_a^b \int_0^{2\pi} \int_{-1}^{1} X_n(r', \theta', \phi') r'^{-n+1} P_n(\cos\gamma) d\mu' d\phi' dr'$$

$$= \frac{4\pi}{2n+1}\left\{CP_n(\mu)\right.$$

$$\left. + \sum_{m=1}^{n} (C_m \cos m\phi + D_m \sin m\phi) T_n^m(\mu)\right\}, \qquad (28)$$

where $C = \int_a^b A r'^{-n+1} dr'$,

$$C_m = \int_a^b A_m r'^{-n+1} dr', \quad D_m = \int_a^b B_m r'^{-n+1} dr'. \qquad (29)$$

Hence, from (21),

$$V_i = 4\pi \sum_{n=0}^{\infty} \frac{r^n}{2n+1} \int_a^b X_n(r', \theta, \phi) r'^{-n+1} dr'. \quad . \qquad (30)$$

Note 3. If ρ is constant, $X_0 = \rho$ and $X_n = 0$ for $n = 1, 2, 3, \ldots$; hence

$$V_e = \frac{M}{r}, \quad V_i = 2\pi\rho(b^2 - a^2).$$

Note 4. If $\rho(r', \theta', \phi')$ is a rational integral function of x', y', z' of degree m, V_i is a rational integral function of x, y, z

of degree m also. For (23) can be derived from (VII, 38), and m is the highest degree of the surface harmonics so obtained. Similarly V_e is a rational integral function of x, y, z of degree m divided by r^{2m+1}.

Finally, the potential at a point within the mass of the shell is obtained by adding the potential V_e for the shell bounded by the spheres of radii a and r to the potential V_i for the shell bounded by the spheres of radii r and b. If this potential is denoted by V_ρ,

$$V_\rho = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \left\{ r^{-n-1} \int_a^r X_n(r', \theta, \phi) r'^{n+2} dr' \right.$$
$$\left. + r^n \int_r^b X_n(r', \theta, \phi) r'^{-n+1} dr' \right\}. \quad (31)$$

Of course V_ρ does not satisfy Laplace's Equation: the reader can easily verify that it is a solution of Poisson's Equation.

Note 5. If ρ is constant,

$$V_\rho = \frac{4\pi}{3} \rho \frac{r^3 - a^3}{r} + 2\pi\rho(b^2 - r^2) = 2\pi\rho\left(b^2 - \frac{1}{3} r^2 - \frac{2}{3} \frac{a^3}{r}\right).$$

The Potential of a Solid Sphere. The expressions for the potential of a solid sphere can be deduced from (27) and (31) by putting a equal to zero.

Density a Rational Integral Function of the Rectangular Coordinates. It was shown in *Note* 4 above, that, when the density is a rational integral function of the rectangular coordinates, so also is V_i. To obtain a formula for V_i we first of all assume that the density $\rho(x', y', z')$ is homogeneous of degree m in x', y', z'; then, by means of (VII., 38) we can put (23) in the form

$$\rho(x', y', z') = Y_m + r'^2 Y_{m-2} + r'^4 Y_{m-4} + \cdots,$$

where Y_n is a solid harmonic of degree n; and so, comparing this expansion with (23), we see that

$$X_n(r', \theta', \phi') = r'^{m-n} Y_n(r', \theta', \phi'),$$

where $m - n$ is even. As $Y_n(r', \theta', \phi')$ is homogeneous of degree n in r', it follows that

$$X_n(r', \theta, \phi) = r'^m r^{-n} Y_n(r, \theta, \phi).$$

Hence, from (30),

$$V_i = 4\pi \sum \frac{1}{2n+1} Y_n(r, \theta, \phi) \int_a^b r'^{m-n+1} dr'$$

$$= 4\pi \sum \frac{1}{2n+1} Y_n(r, \theta, \phi) \frac{b^{m-n+2} - a^{m-n+2}}{m-n+2}, \qquad (32)$$

where $n = m, m-2, m-4, \ldots$

Similarly, from (27),

$$V_e = 4\pi \sum \frac{1}{2n+1} \frac{Y_n(r, \theta, \phi)}{r^{2n+1}} \frac{b^{m+n+3} - a^{m+n+3}}{m+n+3}. \qquad (33)$$

If $\rho(x', y', z')$ is not homogeneous in x', y', z', the values of V_i and V_e are obtained for each homogeneous part of ρ, and then added.

Example. If $\rho = ax' + by' + cz'$, show that

$$V_i = \frac{2\pi}{3}(b^2 - a^2)(ax + by + cz),$$

$$V_e = \frac{4\pi}{15}(b^5 - a^5)\frac{ax + by + cz}{r^3}.$$

§ 7. **Equivalent Distributions of Masses.** It will now be proved that there is one and only one surface distribution of mass $\sigma(\theta, \phi)$ on a thin shell $r = b$ which has the same effect at points external to the shell as a given distribution of density $\rho(r, \theta, \phi)$, in a thick shell (or solid sphere) bounded externally by the sphere $r = b$.

Let V_e and W_e be the potentials of the thin and thick shells respectively: since their effects at external points are the same, it follows that

$$\frac{\partial V_e}{\partial x} = \frac{\partial W_e}{\partial x}, \quad \frac{\partial V_e}{\partial y} = \frac{\partial W_e}{\partial y}, \quad \frac{\partial V_e}{\partial z} = \frac{\partial W_e}{\partial z},$$

so that

$$V_e = W_e + C,$$

where C is a constant. Since V_e and W_e both vanish at infinity, this constant must be zero, and therefore V_e and W_e are equal.

By comparing (27) and (8) we see that, as these potentials are equal for all values of r greater than b,

$$Z_n(\theta, \phi) = \int_a^b X_n(r', \theta, \phi)\left(\frac{r'}{b}\right)^{n+2} dr',$$

where

$$\rho(r', \theta', \phi') = \sum_{n=0}^{\infty} X_n(r', \theta', \phi'), \quad \sigma(\theta', \phi') = \sum_{n=0}^{\infty} Z_n(\theta', \phi').$$

Thus $\sigma(\theta', \phi')$ is uniquely determined when $\rho(r', \theta', \phi')$ is given.

Note 1. The converse theorem does not hold.

Note 2. As the first terms in both expansions, (27) and (8), are of the form M/r, the total mass is the same in both cases.

It can be shown in exactly the same way that for a thick shell bounded by the spheres $r = a$, $r = b$, where $a < b$, a surface distribution over the sphere $r = a$ can be found which is equivalent at all points within the hollow to the mass distribution of the thick shell. In this case, however, C is not necessarily zero, so that the two masses need not be equal. This is due to the fact that a uniform distribution of mass on a spherical shell gives rise to no forces within the shell [cf. § 6, *Note* 3].

§ 8. **The First Boundary Problem for the Sphere.** If the value of the potential is given at all points of the boundary of a region, the problem of determining the value of the potential at all points of the region is known as the First Boundary Problem. The solutions of this problem for a region bounded by a thin spherical shell are given in formulæ (12) and (13). For a thick spherical shell the values of V_i and V_e are obtained in the same way.

Thus, if, at the external boundary $r = b$,

$$V = f(\theta, \phi) = \sum_{n=0}^{\infty} Y_n(\theta, \phi), \qquad (34)$$

then

$$V_e = \sum_{n=0}^{\infty} \left(\frac{b}{r}\right)^{n+1} Y_n(\theta, \phi); \qquad (35)$$

while, if, at the internal boundary $r = a$,

$$V = F(\theta, \phi) = \sum_{n=0}^{\infty} Z_n(\theta, \phi), \qquad (36)$$

then

$$V_i = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^{n} Z_n(\theta, \phi). \qquad (37)$$

Example. If V has the constant value C on the sphere $r = a$, $V_i = C$.

Poisson's Integrals. As in § 5 it follows that

$$V_i = \frac{a(a^2 - r^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{F(\theta', \phi')d\mu' d\phi'}{(r^2 - 2ar\cos\gamma + a^2)^{\frac{3}{2}}}, \qquad (38)$$

$$V_e = \frac{b(r^2 - b^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{f(\theta', \phi')d\mu' d\phi'}{(r^2 - 2br\cos\gamma + b^2)^{\frac{3}{2}}}. \qquad (39)$$

Space bounded by Concentric Spheres. Again, let V be the potential in a region bounded by the spheres $r = a$, $r = b$, where $a < b$, and unoccupied by attracting matter, and let the values of V on $r = a$ and $r = b$ be

$$F(\theta, \phi) = \sum_{n=0}^{\infty} Z_n(\theta, \phi), \quad . \quad . \quad . \quad (40)$$

and

$$f(\theta, \phi) = \sum_{n=0}^{\infty} Y_n(\theta, \phi) \quad . \quad . \quad . \quad (41)$$

respectively. In the region $a \leqq r \leqq b$, V can be expanded in the form

$$V = \sum_{n=0}^{\infty} r^n X_n(\theta, \phi) + \sum_{n=0}^{\infty} r^{-n-1} X'_n(\theta, \phi), \quad . \quad (42)$$

the first part of the expression being due to the matter external to the sphere $r = b$, and the second part to the matter within the sphere $r = a$.

When $r = b$,

$$V = \sum_{n=0}^{\infty} Y_n(\theta, \phi) = \sum_{n=0}^{\infty} b^n X_n(\theta, \phi) + \sum_{n=0}^{\infty} b^{-n-1} X'_n(\theta, \phi).$$

and, when $r = a$,

$$V = \sum_{n=0}^{\infty} Z_n(\theta, \phi) = \sum_{n=0}^{\infty} a^n X_n(\theta, \phi) + \sum_{n=0}^{\infty} a^{-n-1} X'_n(\theta, \phi).$$

It follows that, for $n = 0, 1, 2, \ldots$,

$$Y_n(\theta, \phi) = b^n X_n(\theta, \phi) + b^{-n-1} X'_n(\theta, \phi),$$
$$Z_n(\theta, \phi) = a^n X_n(\theta, \phi) + a^{-n-1} X'_n(\theta, \phi);$$

and, by solving these equations for $X_n(\theta, \phi)$ and $X'_n(\theta, \phi)$, and substituting in (42), we find that

$$V = \sum_{n=0}^{\infty} \frac{(r^n a^{-n-1} - a^n r^{-n-1}) Y_n(\theta, \phi) + (b^n r^{-n-1} - r^n b^{-n-1}) Z_n(\theta, \phi)}{b^n a^{-n-1} - a^n b^{-n-1}}. \qquad (43)$$

Example. If the values of the potential on the spheres $r = a$ and $r = b$ are constant and equal to A and B respectively, show that, for $a \leq r \leq b$,

$$V = \frac{Bb}{r}\frac{r-a}{b-a} + \frac{Aa}{r}\frac{b-r}{b-a}.$$

§ 9. **The Second Boundary Problem for the Sphere.** It may be that, instead of being given the value of the potential on the surface of a region, we are given the partial derivative of the potential with regard to the normal to the surface (*i.e.* the component of the force along the normal to the surface); it will be assumed that the normal is measured away from the region. The problem of finding the potential at all points within the region is called the Second Boundary Problem.

First of all, consider the potential V_e at points external to the sphere $r = b$; the derivative of V_e with regard to the normal to the surface of this exterior space is given to be $f(\theta, \phi)$, so that

$$\left[-\frac{\partial V_e}{\partial r}\right]_{r=b} = f(\theta, \phi),$$

and V_e satisfies Laplace's Equation at all points outside the sphere.

We assume that V_e can be expanded in the form

$$V_e = \sum_{n=0}^{\infty} r^{-n-1} X_n(\theta, \phi),$$

so that

$$\left[-\frac{\partial V_e}{\partial r}\right]_{r=b} = \sum_{n=0}^{\infty} \frac{n+1}{b^{n+2}} X_n(\theta, \phi).$$

Then, if

$$f(\theta, \phi) = \sum_{n=0}^{\infty} Y_n(\theta, \phi) \qquad . \quad . \quad . \quad (44)$$

$$Y_n(\theta, \phi) = \frac{n+1}{b^{n+2}} X_n(\theta, \phi),$$

and therefore

$$V_e' = b\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{b}{r}\right)^{n+1}Y_n(\theta, \phi). \qquad (45)$$

As this potential satisfies the given conditions it must (§ 1, *Theorem* 2, *Note*) be the required solution.

Next, let V_i be a function which satisfies Laplace's Equation within the sphere $r = a$ and satisfies the equation

$$\left[\frac{\partial V_i}{\partial r}\right]_{r=a} = F(\theta, \phi)$$

on the sphere, where $F(\theta, \phi)$ is a given function. If

$$F(\theta, \phi) = \sum_{n=0}^{\infty} Z_n(\theta, \phi)$$

it can be shown that the value of $Z_0(\theta, \phi)$ must be zero. For (VIII., 7) the surface integral

$$-\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\left[\frac{\partial V_i}{\partial r}\right]_{r=a}d\mu d\phi = -\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}F(\theta, \phi)d\mu d\phi$$

is equal to the total attracting mass within the sphere, which in this case is zero: hence

$$\int_0^{2\pi}\int_{-1}^{1}\sum_{n=0}^{\infty}Z_n(\theta, \phi)d\mu d\phi = 0.$$

But, if $n = 1, 2, 3, \ldots$

$$\int_0^{2\pi}\int_{-1}^{1}Z_n(\theta, \phi)d\mu d\phi = 0;$$

and, since $Z_0(\theta, \phi)$ is a constant,

$$\int_0^{2\pi}\int_{-1}^{1}Z_0(\theta, \phi)d\mu d\phi = 4\pi Z_0(\theta, \phi).$$

Hence $Z_0(\theta, \phi)$ is zero, and

$$F(\theta, \phi) = \sum_{n=1}^{\infty}Z_n(\theta, \phi). \qquad (46)$$

We now assume that

$$V_i = \sum_{n=0}^{\infty} r^n X_n(\theta, \phi),$$

which gives

$$\left[\frac{\partial V_i}{\partial r}\right]_{r=a} = \sum_{n=1}^{\infty} na^{n-1} X_n(\theta, \phi).$$

As this is equal to $F(\theta, \phi)$, it follows from (46) that

$$Z_n(\theta, \phi) = na^{n-1} X_n(\theta, \phi), \ n = 1, 2, 3, \dots$$

Hence

$$V_i = C + a \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n Z_n(\theta, \phi), \quad \cdot \quad \cdot \quad (47)$$

where C is the value of X_0, and is the constant of § 1, Theorem 2, Note.

§ 10. **Potential of a Homogeneous Spheroid.** A solid body which differs very little in shape from a sphere is called a spheroid. The equation of the surface of such a body may be put in the form

$$r = a\{1 + \alpha f(\theta, \phi)\}, \qquad (48)$$

where α is a small quantity; it will here be assumed that α is so small that its square and higher powers may be neglected.

Theorem 1. If the density ρ of the spheroid is uniform, its potential is equal to the sum of the potentials of the sphere bounded by the surface $r = a$ and of a thin spherical shell of radius a and surface-density $\sigma = \rho a \alpha f(\theta, \phi)$.

For, if V is the potential at the point $P(r, \theta, \phi)$, and $Q_1(r', \theta', \phi')$ is any point of the spheroid,

$$V = \int_0^{2\pi}\int_{-1}^{1}\int_0^{r_1} \frac{\rho r'^2 dr' d\mu' d\phi'}{R_1},$$

where $PQ_1 = R_1$ and $r_1 = a\{1 + \alpha f(\theta', \phi')\}$. Then

$$V = V_1 + V_2,$$

where

$$V_1 = \int_0^{2\pi}\int_{-1}^{1}\int_0^{a} \frac{\rho r'^2 dr' d\mu' d\psi'}{R_1}$$

is the potential of the sphere, and

$$V_2 = \int_0^{2\pi}\int_{-1}^{1}\int_a^{r_1}\frac{\rho r'^2 dr' d\mu' d\phi'}{R_1}.$$

If P lies outside the sphere

$$V_2 = \int_0^{2\pi}\int_{-1}^{1}\int_a^{r_1}\rho r'^2\left\{\sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n(\cos\gamma)\right\}dr' d\mu' d\phi'$$

$$= \int_0^{2\pi}\int_{-1}^{1}\rho\left\{\sum_{n=0}^{\infty}\frac{r_1^{n+3}-a^{n+3}}{n+3}\,\frac{P_n(\cos\gamma)}{r^{n+1}}\right\}d\mu' d\phi'.$$

Here expand $\qquad r_1^{n+3} = a^{n+3}\{1 + \alpha f(\theta', \phi')\}^{n+3}$

in powers of α; then, neglecting α^2 and higher powers of α, we have

$$V_2 = \int_0^{2\pi}\int_{-1}^{1}\rho\left\{\sum_{n=0}^{\infty}a^{n+3}\alpha f(\theta', \phi')\frac{P_n(\cos\gamma)}{r^{n+1}}\right\}d\mu' d\phi'$$

$$= \int_0^{2\pi}\int_{-1}^{1}\sigma\left\{\sum_{n=0}^{\infty}\frac{a^n}{r^{n+1}}P_n(\cos\gamma)\right\}a^2 d\mu' d\phi'$$

$$= \int_0^{2\pi}\int_{-1}^{1}\frac{\sigma a^2 d\mu' d\phi'}{R},$$

where R is the distance of P from the point (a, θ', ϕ') on the sphere $r = a$. This is the potential (1) of the thin spherical shell of density σ. By expanding R_1 in ascending powers of r the same result can be obtained for points within the sphere $r = a$.

Corollary 1. If $f(\theta, \phi) = \sum_{n=0}^{\infty} Y_n(\theta, \phi)$

the potential at points external to the spheroid is, by (8),

$$V_e = \frac{4}{3}\frac{\pi\rho a^3}{r} + 4\pi\rho a^2\alpha\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{a}{r}\right)^{n+1}Y_n(\theta, \phi). \tag{49}$$

Corollary 2. The potential at an internal point is, by (9) and § 6, *Note* 5,

$$V_p = 2\pi\rho(a^2 - \tfrac{1}{3}r^2) + 4\pi\rho a^2\alpha\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{r}{a}\right)^n Y_n(\theta, \phi). \tag{50}$$

Example 1. If the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a spheroid, express the radius r in terms of spherical harmonics.

[Put $a = c(1 + \alpha)$, $b = c(1 + \beta)$, where the squares of α and β can be neglected: then show that (cf. Ch. VII., *ex.* 5)

$$\begin{aligned} r &= c\{1 + \alpha\sin^2\theta\cos^2\phi + \beta\sin^2\theta\sin^2\phi\} \\ &= c\{1 + \tfrac{1}{3}(\alpha + \beta) - \tfrac{1}{3}(\alpha + \beta)\,\mathrm{P}_2(\cos\theta) + \tfrac{1}{6}(\alpha - \beta)\cos 2\phi\,\mathrm{T}_2^2(\cos\theta).] \end{aligned}$$

Theorem 2. If M is the mass of the spheroid bounded by

$$r = a\left\{1 + \alpha\sum_{n=0}^{\infty}\mathrm{Y}_n(\theta, \phi)\right\},$$

then

$$\mathrm{M} = \tfrac{4}{3}\pi\rho a^3\{1 + 3\alpha\mathrm{Y}_0\}. \qquad (51)$$

For

$$\begin{aligned} \mathrm{M} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^{r}\rho r'^2 dr'd\mu'd\phi' \\ &= \int_0^{2\pi}\int_{-1}^{1}\tfrac{1}{3}\rho r^3 d\mu'd\phi' \\ &= \int_0^{2\pi}\int_{-1}^{1}\tfrac{1}{3}\rho a^3\left\{1 + 3\alpha\sum_{n=0}^{\infty}\mathrm{Y}_n(\theta', \phi')\right\}d\mu'd\phi' \\ &= \tfrac{4}{3}\pi\rho a^3\{1 + 3\alpha\mathrm{Y}_0\}. \end{aligned}$$

Corollary. From (49) it follows that

$$\mathrm{V}_e = \frac{\mathrm{M}}{r} + 4\pi\rho a^2\alpha\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\frac{a}{r}\right)^{n+1}\mathrm{Y}_n(\theta, \phi). \qquad (52)$$

Example 2. For an oblate ellipsoid of revolution put $\beta = \alpha$ in *ex.* 1, and get

$$r = c\{1 + \tfrac{2}{3}\alpha - \tfrac{2}{3}\alpha\mathrm{P}_2(\cos\theta)\}.$$

Then, from (52) and (51),

$$\mathrm{V}_e = \frac{\mathrm{M}}{r} - \frac{8}{3}\pi\rho c^2\alpha\frac{1}{5}\left(\frac{c}{r}\right)^3\mathrm{P}_2(\cos\theta) = \frac{\mathrm{M}}{r} - \frac{2}{5}\alpha\mathrm{M}\frac{c^2}{r^3}\mathrm{P}_2(\cos\theta).$$

Theorem 3. If the origin is the centroid of the spheroid bounded by

$$r = a\left\{1 + \alpha\sum_{n=0}^{\infty}\mathrm{Y}_n(\theta, \phi)\right\},$$

the term $\mathrm{Y}_1(\theta, \phi)$ must vanish.

For, if (x, y, z) is the centroid,

$$\begin{aligned} \mathrm{M}x &= \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{r} r'^2(r' \sin\theta' \cos\phi') \sin\theta' \, dr'd\theta'd\phi' \\ &= \rho\int_0^{2\pi}\int_0^{\pi} \tfrac{1}{4}r^4(\sin\theta' \cos\phi') \sin\theta' \, d\theta'd\phi' \end{aligned}$$

$$= \tfrac{1}{4}\rho a^4\int_0^{2\pi}\int_0^{\pi}\left\{1 + 4\alpha\sum_{n=0}^{\infty} \mathrm{Y}_n(\theta', \phi')\right\}(\sin\theta' \cos\phi') \sin\theta' \, d\theta'd\phi'$$

$$= \rho a^4\alpha\int_0^{2\pi}\int_0^{\pi} \mathrm{Y}_1(\theta', \phi')(\sin\theta' \cos\phi') \sin\theta' \, d\theta'd\phi',$$

since $\sin\theta' \cos\phi'$ is the surface harmonic $\cos\phi' \, \mathrm{T}_1^1 (\cos\theta')$. Therefore, since $x = 0$,

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{Y}_1(\theta', \phi')(\sin\theta' \cos\phi') \sin\theta' \, d\theta'd\phi' = 0. \quad . \quad (53)$$

Similarly, since y and z both vanish,

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{Y}_1(\theta', \phi')(\sin\theta' \sin\phi') \sin\theta' \, d\theta'd\phi' = 0, \quad . \quad (54)$$

and

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{Y}_1(\theta', \phi')(\cos\theta') \sin\theta' \, d\theta'd\phi' = 0. \quad . \quad (55)$$

Now $\mathrm{Y}_1(\theta', \phi')$ is of the form

$$\mathrm{A} \cos\theta' + \mathrm{B} \sin\theta' \cos\phi' + \mathrm{C} \sin\theta' \sin\phi',$$

where A, B, C are constants. From the equations (53), (54) and (55) it follows that A, B, and C all vanish. This may be shown either by direct integration, or by multiplying (55), (53) and (54) by A, B, and C respectively and adding; this gives

$$\int_0^{2\pi}\int_0^{\pi} \{\mathrm{Y}_1(\theta', \phi')\}^2 \sin\theta' \, d\theta'd\phi' = 0,$$

from which, as the integrand cannot be negative, it follows that $\mathrm{Y}_1(\theta, \phi)$ is zero for all values of θ and ϕ on the sphere, and therefore vanishes identically.

§ 11. **Potential of a Heterogeneous Spheroid.** If the spheroid is not homogeneous, but consists of strata of different densities such that the surfaces of equal density differ but little from a sphere, the potential can be deduced from the results of the previous section. Let any surface of equal density be

$$r = a'\{1 + \alpha f(\theta, \phi, a')\}, \quad . \quad . \quad . \quad (56)$$

where α is so small that its square and higher powers may be neglected, and a' is the parameter of surfaces of equal density. The external boundary of the spheroid is the surface of equal density whose parameter is a; the density ρ is a function $F(a')$ of a' alone, and

$$f(\theta, \phi, a') = \sum_{n=0}^{\infty} Y_n(\theta, \phi, a').$$

When the surfaces of equal density are all similar, $f(\theta, \phi, a')$ and $Y_n(\theta, \phi, a')$ are independent of a'.

If now $V(a')$ is the potential of that portion of the spheroid which is bounded externally by the surface (56), $\frac{dV(a')}{da'}$ measures the rate at which $V(a')$ varies with a'. But if the spheroid bounded by this surface were homogeneous, of density $\rho = F(a')$, the potential would be, from (49), for an external point,

$$U_e(a') = \frac{4}{3}\frac{\pi\rho a'^3}{r} + 4\pi\rho a'^2\alpha \sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{a'}{r}\right)^{n+1} Y_n(\theta, \phi, a');$$

and, from (50), for an internal point,

$$U_\rho(a') = 2\pi\rho(a'^2 - \frac{1}{3}r^2) + 4\pi\rho a'^2\alpha \sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{r}{a'}\right)^{n} Y_n(\theta, \phi, a').$$

Since the density in this case is that of the stratum a', the derivatives of these functions with regard to a', ρ being treated as a constant, give the corresponding values of $\frac{dV(a')}{da'}$; so that, for external and internal points respectively,

$$\frac{dV_e(a')}{da'} = \rho\frac{\partial}{\partial a'}\left\{\frac{1}{\rho}U_e(a')\right\}, \quad \frac{dV_i(a')}{da'} = \rho\frac{\partial}{\partial a'}\left\{\frac{1}{\rho}U_\rho(a')\right\}.$$

Hence, for the complete spheroid, the potential for external points is,

$$V_e = \int_0^a \rho\frac{\partial}{\partial a'}\left\{\frac{4}{3}\frac{\pi a'^3}{r} + 4\pi a'^2\alpha\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{a'}{r}\right)^{n+1} Y_n(\theta, \phi, a')\right\}da'$$

$$= \frac{M}{r} + 4\pi\alpha\int_0^a \rho\frac{\partial}{\partial a'}\left\{\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{a'^{n+3}}{r^{n+1}}Y_n(\theta, \phi, a')\right\}da'. \quad (57)$$

Spheroidal Shell. For a heterogeneous spheroidal shell bounded by surfaces of equal density at which $a' = a$ and $a' = b$, where $a < b$, the limits of integration are taken to be a and b in place of 0 and a. At an internal point,

$$V_i = 4\pi\int_a^b \rho a' da' + 4\pi\alpha\int_a^b \rho \frac{\partial}{\partial a'}\left\{\sum_{n=0}^{\infty} \frac{1}{2n+1}\frac{r^n}{a'^{n-2}} Y_n(\theta, \phi, a')\right\} da'. \tag{58}$$

For a point in the body of the shell on the surface whose parameter is a', the potential is

$$\begin{aligned} V_\rho = {} & 4\pi\int_a^{a'} \rho\left[\frac{a'^2}{r} + \alpha\sum_{n=0}^{\infty} \frac{1}{r^{n+1}}\frac{1}{2n+1}\frac{\partial}{\partial a'}\{a'^{n+3} Y_n(\theta, \phi, a')\}\right] da' \\ & + 4\pi\int_{a'}^{b} \rho\left[a' + \alpha\sum_{n=0}^{\infty} \frac{r^n}{2n+1}\frac{\partial}{\partial a'}\{a'^{-n+2} Y_n(\theta, \phi, a')\}\right] da'. \end{aligned} \tag{59}$$

When the integrations have been carried out, V_ρ is expressed as a function of r, θ, ϕ, and a'. In differentiating partially with regard to r, θ, ϕ in order to obtain the corresponding components of attraction it should be borne in mind that a' is a function of these co-ordinates. It can, however, be shown that the value of $\frac{\partial V_\rho}{\partial a'}$ is zero, so that the correct values of $\frac{\partial V_\rho}{\partial r}$, $\frac{\partial V_\rho}{\partial \theta}$, $\frac{\partial V_\rho}{\partial \phi}$ are obtained if a' is treated as a constant. The proof is as follows:—

$$\begin{aligned} \frac{\partial V_\rho}{\partial a'} = 4\pi\rho\Bigg[& \frac{a'^2}{r} + \alpha\sum_{n=0}^{\infty}\left\{\frac{a'^{n+2}}{r^{n+1}}\frac{n+3}{2n+1}Y_n(\theta, \phi, a') \right. \\ & \qquad\qquad \left. + \frac{a'^{n+3}}{r^{n+1}}\frac{1}{2n+1}\frac{\partial}{\partial a'}Y_n(\theta, \phi, a')\right\} \\ & - a' + \alpha\sum_{n=0}^{\infty}\left\{\frac{r^n}{a'^{n-1}}\frac{n-2}{2n+1}Y_n(\theta, \phi, a') \right. \\ & \qquad\qquad \left. - \frac{r^n}{a'^{n-2}}\frac{1}{2n+1}\frac{\partial}{\partial a'}Y_n(\theta, \phi, a')\right\}\Bigg] \end{aligned}$$

In the term $\frac{a'^2}{r}$ put $r = a'\left\{1 + \alpha\sum_{n=0} Y_n(\theta, \phi, a')\right\}$; then

$$\frac{a'^2}{r} = a'\left\{1 - \alpha\sum_{n=0}^{\infty} Y_n(\theta, \phi, a')\right\},$$

powers of α above the first being neglected. In the terms of which α is a factor we may put $r = a'$, since any closer approximation would introduce powers of α above the first: thus

$$\begin{aligned}\frac{\partial V_\rho}{\partial a'} = 4\pi\rho\Bigg[\, & a' - a'\alpha\sum_{n=0}^{\infty} Y_n(\theta, \phi, a') \\ & + \alpha\sum_{n=0}^{\infty}\left\{a'\frac{n+3}{2n+1}Y_n(\theta, \phi, a') + \frac{a'^2}{2n+1}\frac{\partial}{\partial a'}Y_n(\theta, \phi, a')\right\} \\ & - a' + \alpha\sum_{n=0}^{\infty}\left\{a'\frac{n-2}{2n+1}Y_n(\theta, \phi, a')\right. \\ & \qquad\qquad\qquad \left. - \frac{a'^2}{2n+1}\frac{\partial}{\partial a'}Y_n(\theta, \phi, a')\right\}\Bigg] \\ = 0.\end{aligned}$$

CHAPTER X

APPLICATIONS TO ELECTROSTATICS

§ 1. **Distribution of Electricity in a Conductor in Electrical Equilibrium.** When a conductor is in a condition of electrical equilibrium, the sum of the electrical forces due to the distribution of electricity within and without the conductor must be zero at all points of the conductor. For, if not, a further movement of electricity in the conductor would take place, and this would contradict the hypothesis of electric equilibrium. Thus, at all points within the conductor the components of force

$$-\frac{\partial V}{\partial x}, \quad -\frac{\partial V}{\partial y}, \quad -\frac{\partial V}{\partial z}$$

are all zero, so that V is constant. Also, throughout the conductor

$$\nabla^2 V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = 0;$$

and therefore, from Poisson's Theorem (VIII., 11) it follows that there is no electricity in the interior of the conductor. Thus the charge of electricity on the conductor lies entirely on the surface. Since the force within the conductor is zero, the density σ of this surface distribution is given by the formula (VIII., 9)—

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n},$$

n being the normal measured away from the conductor. The force at the surface must be in the direction of the normal, for if there were a tangential component of force a movement of electricity would take place along the surface of the conductor. The potential V, due to charges on and outside the conductor, is therefore constant throughout the conductor. For a system of conductors this must be true for each conductor, but the potentials of the different conductors may differ from each other.

§ 2. **Distribution of Electricity on an Insulated Conducting Sphere.** Let a quantity M of free electricity be communicated to an insulated sphere whose surface is $r = a$, and let the sphere be subjected to electrical forces arising from a distribution of electricity external to the sphere. From Ch. VIII., § 4, Cor. 2 it follows that the potential U of this external distribution can, within and on the surface of the sphere, be expanded in the form

$$U = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n Y_n(\theta, \phi). \quad . \quad . \quad . \quad (1)$$

If, moreover, the surface distribution σ on the sphere be expressed in the form

$$\sigma = \sum_{n=0}^{\infty} Z_n(\theta, \phi), \quad . \quad . \quad . \quad (2)$$

the potential V of this distribution is, by (IX., 9),

$$V = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{r^n}{a^{n-1}} Z_n(\theta, \phi). \quad . \quad . \quad (3)$$

Now within and on the sphere the potential has a constant value, C say, so that

$$U + V = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \left\{ Y_n(\theta, \phi) + \frac{4\pi a}{2n+1} Z_n(\theta, \phi) \right\} = C \quad (4)$$

at all points of the sphere. From this equation we deduce that

$$Y_n(\theta, \phi) + \frac{4\pi a}{2n+1} Z_n(\theta, \phi) = 0, \quad . \quad . \quad (5)$$

for $n = 1, 2, 3, \ldots$, and that

$$Y_0(\theta, \phi) + 4\pi a Z_0(\theta, \phi) = C.$$

By means of (5) all the Z's except Z_0 can be expressed in terms of the Y's; as for Z_0, from Ch. IX., § 2, Note 1, the first term in (3) is M/a, so that

$$Z_0(\theta, \phi) = \frac{M}{4\pi a^2}.$$

The constant potential Y_0 gives rise to no forces, and has therefore no effect on the distribution of the electricity on the conductor.

Thus, finally,

$$\sigma = \frac{M}{4\pi a^2} - \frac{1}{4\pi a}\sum_{n=1}^{\infty}(2n+1)Y_n(\theta, \phi). \quad . \quad . \quad (6)$$

This density σ is made up of two parts: firstly, the uniform distribution $M/(4\pi a^2)$ which would be the density of the charge M on the sphere if no external forces were present; and, secondly, the density

$$-\frac{1}{4\pi a}\sum_{n=1}^{\infty}(2n+1)Y_n(\theta, \phi)$$

induced by the external forces. This would be the actual distribution if no free electricity were communicated to the conductor.

These results can also be obtained as follows. Within and on the sphere the potential of the surface distribution whose density is to be determined is, from (1),

$$V_i = C - \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n Y_n(\theta, \phi); \quad . \quad . \quad . \quad (7)$$

and, from the value of this potential on the sphere $r = a$ the corresponding external potential is found (IX., 13) to be

$$V_e = \{C - Y_0(\theta, \phi)\}\frac{a}{r} - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} Y_n(\theta, \phi). \quad (8)$$

Now, if $\bar{V}_e$ and $\bar{V}_i$ are the total external and internal potentials,

$$\sigma = -\frac{1}{4\pi}\left[\frac{\partial \bar{V}_e}{\partial r} - \frac{\partial \bar{V}_i}{\partial r}\right]_{r=a};$$

but the potential of the external distribution makes no contribution to σ, since its partial derivative with regard to r is continuous at the surface of the sphere. Hence

$$\sigma = -\frac{1}{4\pi}\left[\frac{\partial V_e}{\partial r} - \frac{\partial V_i}{\partial r}\right]_{r=a},$$

where V_e and V_i are given by (8) and (7): this formula leads to (6), provided only that we note that, since $\lim\limits_{r \to \infty} (rV_e) = M$,

$$\{C - Y_0(\theta, \phi)\}a = M.$$

§ 3. **Insulated Sphere and External Point-Charge.** Suppose that a charge M of free electricity has been communicated to an insulated spherical conductor bounded by the sphere $r = a$, and that, at an external point $A(c, \theta', \phi')$, where $c > a$, there is a point-charge of strength m. Then, at the point (r, θ, ϕ), where $r < c$, (1) becomes

$$U = \frac{m}{\sqrt{(c^2 - 2cr\cos\gamma + r^2)}} = \frac{m}{c}\sum_{n=0}^{\infty}\left(\frac{r}{c}\right)^n P_n(\cos\gamma), \qquad (9)$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi');$$

hence, from (6),

$$\sigma = \frac{M}{4\pi a^2} - \frac{m}{4\pi ac}\sum_{n=1}^{\infty}(2n + 1)\left(\frac{a}{c}\right)^n P_n(\cos\gamma) \quad . \quad . \quad (10)$$

$$= \frac{M}{4\pi a^2} + \frac{m}{4\pi ac}\left\{1 - \frac{c(c^2 - a^2)}{(c^2 - 2ca\cos\gamma + a^2)^{\frac{3}{2}}}\right\}, \text{ by V., 38,}$$

$$= \frac{M}{4\pi a^2} + \frac{m}{4\pi ac}\left\{1 - \frac{ct^2}{R^3}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where t is the length of the tangent from A to the sphere and R is the distance of the point $P(a, \theta, \phi)$ on the surface of the sphere from A.

If σ' denotes the part of this surface-density which is induced by the point-charge at A,

$$\sigma' = \frac{m}{4\pi ac}\left\{1 - \frac{ct^2}{R^3}\right\}. \quad . \quad . \quad . \quad . \quad (12)$$

By means of this formula the distribution of the induced electricity over the sphere can be ascertained. If AO cuts the surface of the sphere in B and B′ (Fig. 18), σ' is constant on those circles of the sphere which lie in planes perpendicular to the diameter BB′. For any point $P(a, \theta, \phi)$ on the sphere γ is the angle AOP, and $R = AP$.

At B, $\gamma = 0$ and $R = c - a$, so that, since $t^2 = c^2 - a^2$,

$$\sigma' = \frac{m}{4\pi ac}\left\{1 - \frac{c(c+a)}{(c-a)^2}\right\},$$

which has the opposite sign from *m*.

At B′, $\gamma = \pi$, $R = c + a$, and

$$\sigma' = \frac{m}{4\pi ac}\left\{1 - \frac{c(c-a)}{(c+a)^2}\right\},$$

which has the same sign as *m*.

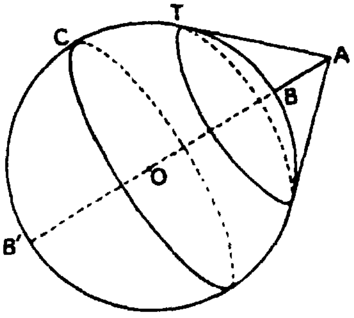

FIG. 16.

If $\gamma = \pi/2$, P is a point such as C on the great circle whose plane is perpendicular to BB′, $R = \sqrt{(c^2 + a^2)}$, and

$$\sigma' = \frac{m}{4\pi ac}\left\{1 - \frac{c(c^2 - a^2)}{(c^2 + a^2)^{\frac{3}{2}}}\right\},$$

so that here σ' has the same sign as *m*.

Again, if P is a point such as T on the circle of contact of the tangents from A to the sphere, $\cos\gamma = a/c$, $R = t$, and

$$\sigma' = \frac{m}{4\pi ac}\left\{1 - \frac{c}{\sqrt{(c^2 - a^2)}}\right\},$$

which has the opposite sign from *m*.

In fact the induced electricity of the opposite sign from *m* gathers on the side of the sphere next to A; that of the same sign as *m* on the side away from A.

To determine the position of the circle which separates the negative from the positive induced electricity, we put $\sigma' = 0$ in (12), and find that $R = c^{\frac{1}{3}}t^{\frac{2}{3}}$, or, if, for this circle, $\gamma = \gamma_1$,

$$c^{\frac{2}{3}}t^{\frac{4}{3}} = c^2 - 2ca\cos\gamma_1 + a^2,$$

so that

$$\cos\gamma_1 = \frac{c^2 + a^2 - c^{\frac{2}{3}}(c^2 - a^2)^{\frac{2}{3}}}{2ac}.$$

The total quantity of induced electricity of the same sign as *m* is

$$\frac{m}{4\pi ac}\int_{\gamma_1}^{\pi}\left\{1 - \frac{c(c^2 - a^2)}{(c^2 - 2ca\cos\gamma + a^2)^{\frac{3}{2}}}\right\}2\pi a^2 \sin\gamma d\gamma$$

$$= \frac{m}{2c}\left[-a\cos\gamma + \frac{c^2 - a^2}{\sqrt{(c^2 - 2ca\cos\gamma + a^2)}}\right]_{\gamma_1}^{\pi}$$

$$= \frac{m}{2c}\left[c + a\cos\gamma_1 - \frac{c^2 - a^2}{c^{\frac{1}{3}}t^{\frac{2}{3}}}\right]$$

$$= \frac{m}{4c^2}\{3c^2 + a^2 - 3c^{\frac{2}{3}}(c^2 - a^2)^{\frac{2}{3}}\}.$$

The potential for internal points of the induced electricity is, by (10) and (IX., 7, 9)

$$V_i = -m\sum_{n=1}^{\infty}\frac{r^n}{c^{n+1}}P_n(\cos\gamma) = \frac{m}{c} - U,$$

so that the constant potential of the sphere when uncharged is $U + V_i = m/c$. For external points, by (IX., 8), the potential of the induced electricity is

$$V_e = -m\sum_{n=1}^{\infty}\frac{a^{2n+1}}{(cr)^{n+1}}P_n(\cos\gamma)$$

$$= \frac{m\,.\,\frac{a}{c}}{r} - \frac{m\frac{a}{c}}{\sqrt{(r^2 - 2rc_0\cos\gamma + c_0^2)}},$$

where $c_0 = a^2/c$; thus

$$V_e = \frac{m\,.\,\frac{a}{c}}{r} - \frac{m\,.\,\frac{a}{c}}{R_0}, \qquad (13)$$

where R_0 is the distance from the point $A_0(c_0, \theta', \phi')$ which is the inverse of A in the sphere. The effect at points external to the sphere of the induced electricity is therefore equivalent to that of two point-charges, the one of strength ma/c at O, and the other of strength $-ma/c$ at A_0.

§ 4. **Distribution of Electricity on an Uninsulated Conducting Sphere.** If the sphere is connected to earth by a conducting wire it is said to be *uninsulated*, and its potential has the value zero. In (4) the value of C is then zero, and (5) holds when $n = 0$, so that (2) becomes

$$\sigma = -\frac{1}{4\pi a}\sum_{n=0}^{\infty}(2n+1)Y_n(\theta, \phi). \qquad (14)$$

The total charge of electricity on the sphere in this case is

$$M' = \int_0^{2\pi}\int_{-1}^{1}\sigma a^2 d\mu d\phi = -aY_0(\theta, \phi). \qquad (15)$$

§ 5. **Uninsulated Sphere and External Point-Charge.** For a point-charge of strength m, at $A(c, \theta', \phi')$, where $c > a$, U has the value (9), and, from (14),

$$\begin{aligned} \sigma &= -\frac{m}{4\pi ac}\sum_{n=0}^{\infty}(2n+1)\left(\frac{a}{c}\right)^n P_n(\cos\gamma) \qquad (16)\\ &= -\frac{m}{4\pi ac}\frac{c(c^2-a^2)}{(c^2-2ca\cos\gamma+a^2)^{\frac{3}{2}}}, \text{ by V., 38,}\\ &= -\frac{mt^2}{4\pi a R^3}. \end{aligned}$$

The induced electricity is therefore all of the opposite sign from that of m, the electricity of the same sign having all been driven to earth. The total charge on the sphere is, by (15), $-ma/c$.

The potential due to this induced electricity at an internal point is, by (IX., 9),

$$V_i = -\frac{m}{c}\sum_{n=0}^{\infty}\left(\frac{r}{c}\right)^n P_n(\cos\gamma) = -U,$$

which, added to the potential U of the inducing charge at A, gives potential zero throughout the sphere.

At external points the corresponding potential is, by (IX., 8),

$$V_e = -m\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(cr)^{n+1}}P_n(\cos\gamma) = -\frac{m\cdot\frac{a}{c}}{R_0},$$

as in (13); thus V_e can be regarded as the potential due to a charge $-ma/c$ at A_0, the inverse of A in the sphere. The effect of the charges m at A and $-ma/c$ at A_0 is thus to give zero potential on the sphere, a result which could be immediately deduced from Ch. IX., § 4, Note 2.

§ 6. **The Green's Function.** The Green's Function for a given region Σ and a point A (the pole) within the region is a function V which satisfies Laplace's Equation at all points P within the region, A excluded, which has the value zero at all points of S, the boundary of Σ, and which is such that $V - 1/R$, where $R = AP$, tends to a definite limit as P tends to A.

V is clearly the potential in the region Σ when the boundary S is uninsulated, and a unit point-charge is situated at A. The corresponding potential when the charge at A is of strength m is mV.

The Green's Function for the Region enclosed by a Sphere. Let the sphere be $r = a$, and let P and A be the points (r, θ, ϕ) and $(c, \theta,' \phi')$, where $c < a$; then

$$R = \sqrt{(r^2 - 2cr\cos\gamma + c^2)},$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$;
so that, if P lies on the sphere,

$$\frac{1}{R} = \sum_{n=0}^{\infty} \frac{c^n}{a^{n+1}} P_n(\cos\gamma). \quad . \quad . \quad . \quad (17)$$

Now the function which satisfies Laplace's Equation within the sphere, and has this value on the surface, is, by (IX., 12),

$$\sum_{n=0}^{\infty} \frac{c^n r^n}{a^{2n+1}} P_n(\cos\gamma) \quad . \quad . \quad . \quad (18)$$

If this function be subtracted from $1/R$, the resulting function

$$V = \frac{1}{R} - \sum_{n=0}^{\infty} \frac{c^n r^n}{a^{2n+1}} P_n(\cos\gamma)$$

vanishes on the surface of the sphere, and is therefore the Green's Function required.

The series (18) is convergent for values of r less than a^2/c, and has the sum

$$\frac{1}{a\sqrt{\left\{1 - 2\frac{rc}{a^2}\cos\gamma + \left(\frac{rc}{a^2}\right)^2\right\}}} = \frac{\frac{a}{c}}{R_0},$$

where $$R_0 = \sqrt{(c_0^2 - 2c_0 r\cos\gamma + r^2)},$$

and $c_0 = a^2/c$: thus R_0 is equal to A_0P, where A_0 is the inverse of A with regard to the sphere

Hence

$$V = \frac{1}{R} - \frac{\frac{a}{c}}{R_0}, \qquad . \quad . \quad . \quad (19)$$

so that the Green's Function is the sum of the potentials of a unit point-charge at A and a point-charge of strength $-a/c$ at A_0. As in the case considered at the end of the previous section, this could be deduced from Ch. IX, § 4, Note 2.

The Green's Function for the Region external to a Sphere. If $A(c, \theta', \phi')$, where $c > a$, is a point external to the sphere $r = a$, then, on the surface of the sphere,

$$\frac{1}{R} = \sum_{n=0}^{\infty} \frac{a^n}{c^{n+1}} P_n(\cos\gamma), \qquad . \quad . \quad . \quad (20)$$

so that the required Green's Function is

$$V = \frac{1}{R} - \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(cr)^{n+1}} P_n(\cos\gamma)$$

$$= \frac{1}{R} - \frac{\frac{a}{c}}{R_0}, \qquad . \quad . \quad . \quad . \quad . \quad (21)$$

where, as before, $R_0 = A_0P$, A_0 being the point inverse to A with regard to the sphere. It will be noticed that the formulae (19) and (21) are identical in form, but that the positions of A and A_0 are interchanged in the two cases. From the equation (21), multiplied by m, the results of § 5 can be deduced.

The Green's Function for the Region bounded by Two Concentric Spheres. Finally, if $A(c, \theta', \phi')$ lies in the region between the spheres $r = a$, $r = b$, where $a < c < b$, then, from (17) and (20) the values of Y_n and Z_n in (IX., 40, 41) are

$$Y_n = \frac{c^n}{b^{n+1}} P_n(\cos\gamma), \quad Z_n = \frac{a^n}{c^{n+1}} P_n(\cos\gamma).$$

By inserting these functions in (IX., 43) we deduce that the Green's Function is

$$V = \frac{1}{R} - \sum_{n=0}^{\infty} \frac{P_n(\cos\gamma)}{b^{2n+1} - a^{2n+1}} \left\{ \frac{c^n}{r^{n+1}} (r^{2n+1} - a^{2n+1}) \right.$$
$$\left. + \frac{a^{2n+1}}{(cr)^{n+1}} (b^{2n+1} - r^{2n+1}) \right\}. \quad (22)$$

§ 7. **Thick Spherical Conducting Shell under the Influence of External Electrical Forces.** Let the shell be bounded by the spheres $r = a$, $r = b$, where $a < b$. The inducing charges lie outside the sphere $r = b$, and therefore, at points within the sphere, the inducing potential U can be expressed in the form (Ch. VIII., § 4, Cor. 2),

$$U = \sum_{n=0}^{\infty} \left(\frac{r}{b}\right)^n Y_n(\theta, \phi). \quad . \quad . \quad . \quad (23)$$

The charge on the shell must lie on the surfaces $r = a$ and $r = b$; let σ be the surface density on $r = b$, and τ that on $r = a$, and let σ and τ be expressed in the forms

$$\sigma = \sum_{n=0}^{\infty} Z_n(\theta, \phi), \quad \tau = \sum_{n=0}^{\infty} X_n(\theta, \phi). \quad . \quad (24)$$

Then, at points throughout the shell, the potentials V and W of these two surface distributions are (IX., 9, 8)

$$V = 4\pi b \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{r}{b}\right)^n Z_n(\theta, \phi), \quad . \quad . \quad (25)$$

$$W = 4\pi a \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{a}{r}\right)^{n+1} X_n(\theta, \phi). \quad . \quad (26)$$

But, throughout the shell, the potential has a constant value C, so that

$$U + V + W = C; \quad (27)$$

therefore, omitting the arguments of X_n, Y_n, and Z_n, we have

$$\sum_{n=0}^{\infty}\left[\left(\frac{r}{b}\right)^n\left\{Y_n + \frac{4\pi b}{2n+1}Z_n\right\} + \left(\frac{a}{r}\right)^{n+1}\frac{4\pi a}{2n+1}X_n\right] = C. \quad (28)$$

Since this equation holds for all values of r between a and b, the coefficients of all powers of r on the left-hand side, except r^0, must vanish. From the vanishing of the coefficients of negative powers of r it results that every X_n is zero, and consequently $\tau = 0$; so that the electrical charge is entirely confined to the outer surface of the conductor. Equation (28) then reduces to (4), with b in place of a. It follows that the electrical behaviour of such a shell subjected to external electrical forces is exactly the same as that of a complete sphere of radius b. [See §§ 2, 3.]

The same holds true if the shell is connected to earth. [See §§ 4, 5.]

§ 8. **Thick Spherical Conducting Shell under the Influence of Internal Electrical Forces.** In the next place, let the inducing electrical forces arise from a distribution of electricity in the hollow space bounded by the interior surface, $r = a$, of the shell. The inducing potential U can then, for points outside this surface, be expressed in the form (VIII., § 4, Cor. 1).

$$U = \sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1}Y_n(\theta, \phi), \quad . \quad . \quad (29)$$

while V and W have the forms given in (25) and (26).

Accordingly, if the constant potential throughout the shell is C,

$$\sum_{n=0}^{\infty}\left[\left(\frac{r}{b}\right)^n\frac{4\pi b}{2n+1}Z_n + \left(\frac{a}{r}\right)^{n+1}\left\{Y_n + \frac{4\pi a}{2n+1}X_n\right\}\right] = C, \quad (30)$$

which gives

$$Z_n = 0, \text{ for } n = 1, 2, 3, \ldots,$$

$$4\pi b Z_0 = C,$$

and

$$X_n = -\frac{2n+1}{4\pi a}Y_n, \text{ for } n = 0, 1, 2, \ldots$$

In this case the inner surface is charged, the density being given by

$$\tau = -\frac{1}{4\pi a}\sum_{n=0}^{\infty}(2n+1)Y_n(\theta, \phi) \quad . \quad . \quad (31)$$

The potential of this charge at points external to the sphere $r = a$ is, by (IX., 8),

$$-\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1}Y_n(\theta, \phi) = -U,$$

so that the distribution on the inner surface cancels the effect of the inducing charges at points outside the sphere $r = a$. The total charge on the inner surface is

$$\int_0^{2\pi}\int_{-1}^{1}\tau a^2 d\mu d\phi = -aY_0,$$

which is equal and opposite in sign to the sum of the inducing charges (*cf.* Ch. VIII., § 2, *Note*).

As the amounts of positive and negative electricity due to induction are equal, the total charge on the outer surface $r = b$ is $M + aY_0$, where M is the free electricity with which the shell has been charged. The density of the distribution on the outer surface is therefore

$$\sigma = \frac{M + aY_0}{4\pi b^2} = Z_0.$$

The induced charge aY_0 on the outer surface acts at external points as if it were a point-charge at the centre; as this charge is equal to the inducing charges in the hollow enclosed by the shell, it follows that the effect of the shell at external points is to cause the inducing charges to act as if they were concentrated at the centre of the shell.

Uninsulated Thick Shell. If the shell is connected to earth, $C = 0$, and therefore $Z_0 = 0$ and $\sigma = 0$. The outer surface is then uncharged, while the distribution on the inner surface is the same as before.

Uninsulated Thick Shell Surrounding a Point-Charge. If there is a point-charge of strength m at $A(c, \theta', \phi')$, where $c < a$, then, if $r > c$,

$$U = \frac{m}{c}\sum_{n=0}^{\infty}\left(\frac{c}{r}\right)^{n+1}P_n(\cos\gamma),$$

so that, in (29),

$$Y_n(\theta, \phi) = \frac{m}{c}\left(\frac{c}{a}\right)^{n+1}P_n(\cos\gamma),$$

and therefore, from (31), the surface density τ on the inner surface is

$$\tau = -\frac{m}{4\pi ac}\sum_{n=0}^{\infty}(2n+1)\left(\frac{c}{a}\right)^{n+1}P_n(\cos\gamma). \quad . \quad (32)$$

This surface distribution, with the point-charge at A, gives potential zero on the sphere $r = a$. The series (32), when summed by (V., 38), gives

$$\tau = -\frac{m}{4\pi a}\frac{a^2 - c^2}{(a^2 - 2ac\cos\gamma + c^2)^{\frac{3}{2}}}. \quad . \quad . \quad (33)$$

Note. The distribution of electricity on a shell bounded by two eccentric spheres, and the distributions on two spheres external to each other, can be deduced from the results obtained in this chapter, by, in each case, inverting the spherical surfaces with regard to a sphere whose centre is at one of the limiting points of the coaxal system to which the spherical surfaces belong. The details of this process will be found in treatises on the Mathematical Theory of Electricity, and will not be given here, as these problems are fully discussed by an alternative method in Chapter XII.

§ 9. **Isolated Conducting Spheroid.** Let the equation of the surface of the spheroid be

$$r' = a\{1 + \alpha F(\theta', \phi')\}, \quad . \quad . \quad . \quad (34)$$

where

$$F(\theta', \phi') = \sum_{n=0}^{\infty} Y_n(\theta', \phi'). \quad . \quad . \quad . \quad (35)$$

Then, if c is the radius of a sphere whose volume is equal to that of the spheroid,

$$c^3 = a^3(1 + 3\alpha Y_0)$$

by (IX., 51). Hence, discarding powers of α above the first, we have

$$c = a(1 + \alpha Y_0) \quad . \quad . \quad . \quad (36)$$

The equation of the surface can therefore be written

$$\begin{aligned} r' &= c\{1 + \alpha F(\theta', \phi') - \alpha Y_0\} \\ &= c\{1 + \alpha f(\theta', \phi')\}, \quad . \quad . \quad . \quad (37) \end{aligned}$$

where
$$f(\theta', \phi') = \sum_{n=1}^{\infty} Y_n(\theta', \phi'). \quad . \quad . \quad (38)$$

We assume that there are no external electrical forces in the field; so that, if σ is the surface density, the potential V at the point (r, θ, ϕ) is given by

$$V = \int_0^{2\pi}\int_{-1}^{1} \frac{\sigma r'^2 d\mu' d\phi'}{\cos\nu \,.\, R}, \quad . \quad . \quad . \quad (39)$$

where ν is the angle between the outward-drawn normal to the surface at the point (r', θ', ϕ') and the radius through that point. The factor $\cos\nu$ can be omitted, as

$$\cos\nu = \left\{1 + \left(\frac{1}{r'}\frac{\partial r'}{\partial\theta'}\right)^2 + \left(\frac{1}{r'\sin\theta'}\frac{\partial r'}{\partial\phi'}\right)^2\right\}^{-\frac{1}{2}},$$

and this differs from unity by a quantity of which α^2 is a factor. At a point (r, θ, ϕ) within the spheroid, $1/R$ can be expanded in ascending powers of r, and

$$V = \sum_{n=0}^{\infty} r^n \int_0^{2\pi}\int_{-1}^{1} \frac{\sigma P_n(\cos\gamma) d\mu' d\phi'}{r'^{n-1}}. \quad . \quad (40)$$

Now, within the conductor V has a constant value C, so that it is independent of r, and consequently the coefficients of r^n in (40), with the exception of r^0, must all vanish; hence

$$\int_0^{2\pi}\int_{-1}^{1} \frac{\sigma P_n(\cos\gamma) d\mu' d\phi'}{r'^{n-1}} = 0, \quad n = 1, 2, 3, \ldots \quad (41)$$

If α were zero, σ would have a constant value $\sigma_0 = M/(4\pi c^2)$, where M is the total charge on the conductor. When α is small we may therefore assume that

$$\sigma = \sigma_0\left\{1 + \alpha \sum_{n=0}^{\infty} Z_n(\theta', \phi')\right\}.$$

But the total charge on the spheroid is

$$M = \int_0^{2\pi}\int_{-1}^{1} \sigma r'^2 d\mu' d\phi' ;$$

and here, using (37) and (38), we may write

$$r'^2 = c^2\left\{1 + \alpha \sum_{n=1}^{\infty} Y_n(\theta', \phi')\right\}^2$$

$$= c^2\left\{1 + 2\alpha \sum_{n=1}^{\infty} Y_n(\theta', \phi')\right\} ;$$

so that

$$\sigma r'^2 = c^2\sigma_0\left\{1 + \alpha \sum_{n=0}^{\infty} Z_n(\theta', \phi') + 2\alpha \sum_{n=1}^{\infty} Y_n(\theta', \phi')\right\}.$$

When this expression is inserted in the integral, it gives

$$M = 4\pi c^2\sigma_0\{1 + \alpha Z_0\} ;$$

and therefore, since $M = 4\pi c^2\sigma_0$, Z_0 must be zero. Thus

$$\sigma = \sigma_0\left\{1 + \alpha \sum_{n=1}^{\infty} Z_n(\theta', \phi')\right\}. \quad . \quad . \quad (42)$$

Again, from (37) and (38) we deduce that

$$\frac{1}{r^{n-1}} = \frac{1}{c^{n-1}}\left\{1 - (n - 1)\alpha \sum_{n=1}^{\infty} Y_n(\theta', \phi')\right\},$$

so that

$$\frac{\sigma}{r'^{n-1}} = \frac{\sigma_0}{c^{n-1}}\left[1 + \alpha \sum_{n=1}^{\infty} \{Z_n(\theta', \phi') - (n - 1)Y_n(\theta', \phi')\}\right].$$

On inserting this expression in (41), we deduce that

$$\frac{4\pi}{2n + 1}\frac{\sigma_0\alpha}{c^{n-1}}\{Z_n(\theta, \phi) - (n - 1)Y_n(\theta, \phi)\} = 0.$$

Hence

$$Z_n(\theta, \phi) = (n - 1)Y_n(\theta, \phi), \quad n = 1, 2, 3, \ldots ;$$

and consequently

$$\sigma = \frac{M}{4\pi c^2}\left\{1 + \alpha \sum_{n=1}^{\infty} (n - 1)Y_n(\theta', \phi')\right\}. \quad . \quad (43)$$

Finally, from (40) and (41), the potential of the conductor is

$$
\begin{aligned}
C &= \int_0^{2\pi}\int_{-1}^{1} r'\sigma d\mu' d\phi' \\
&= \int_0^{2\pi}\int_{-1}^{1} c\sigma_0\Big[1 + \alpha\sum_{n=1}^{\infty}\{Z_n(\theta', \phi') + Y_n(\theta', \phi')\}\Big]d\mu' d\phi' \\
&= 4\pi c\sigma_0 = \frac{M}{c}. \qquad \qquad (44)
\end{aligned}
$$

Example. Take the oblate ellipsoid of revolution

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{a^2(1 - \alpha)^2} = 1,$$

where α is so small that its square and higher powers may be neglected. The polar equation of the surface may be written

$$
\begin{aligned}
r' &= a(1 - \alpha \cos^2\theta') = a[1 - \alpha\{\tfrac{1}{3} + \tfrac{2}{3}P_2(\cos\theta')\}] \\
&= c\{1 - \alpha \,.\, \tfrac{2}{3}P_2(\cos\theta')\},
\end{aligned}
$$

where $c = a(1 - \frac{1}{3}\alpha)$ Hence, from (43), (37), and (38),

$$
\begin{aligned}
\sigma &= \frac{M}{4\pi c^2}\Big\{1 - \alpha \,.\, \frac{2}{3}P_2(\cos\theta')\Big\} \\
&= \frac{M}{4\pi a^2}\Big\{1 + \frac{2}{3}\alpha - \frac{2}{3}\alpha\frac{3\cos^2\theta' - 1}{2}\Big\} \\
&= \frac{M}{4\pi a^2}(1 + \alpha\sin^2\theta')
\end{aligned}
$$

CHAPTER XI

ELLIPSOIDS OF REVOLUTION

§ 1. Transformation of Laplace's and Poisson's Equations to an Orthogonal System of Curvilinear Co-ordinates. Let the Cartesian co-ordinates x, y, z be expressed in terms of new variables λ, μ, ν by the equations

$$x = f(\lambda, \mu, \nu), \quad y = \phi(\lambda, \mu, \nu), \quad z = \psi(\lambda, \mu, \nu). \qquad (1)$$

If the variable λ has a constant value assigned to it, the point (x, y, z) will lie on a fixed surface whose equation is obtained by eliminating μ and ν between the three equations (1). For varying values of λ we thus obtain an infinite system of such surfaces, with λ as parameter, and there are two corresponding systems for the parameters μ and ν. If the surfaces whose parameters are λ, μ, and ν intersect in the point (x, y, z), λ, μ, and ν are called the *Curvilinear Co-ordinates* of the point. The polar co-ordinates r, θ, ϕ constitute such a system of curvilinear co-ordinates. If, as in the case of polar co-ordinates, the surfaces of the different systems intersect at right angles, the system of co-ordinates is said to be *Orthogonal*.

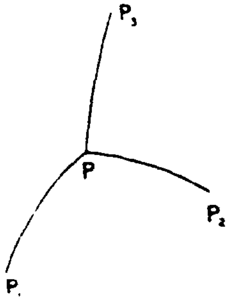

FIG. 19.

Now let the three surfaces which pass through a point P (Fig. 19) intersect along the curves PP_1, PP_2, and PP_3: the surfaces P_2PP_3, P_3PP_1, P_1PP_2 belong to the λ, μ, and ν systems respectively. Along PP_1 μ and ν are constant, so that the direction-cosines of the tangent at P to PP_1 are proportional to $\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}, \frac{\partial z}{\partial \lambda}$; hence their values are

$$\frac{1}{l}\frac{\partial x}{\partial \lambda}, \quad \frac{1}{l}\frac{\partial y}{\partial \lambda}, \quad \frac{1}{l}\frac{\partial z}{\partial \lambda},$$

where
$$l = \sqrt{\left\{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2\right\}}.$$

Similarly, if (2)
$$m = \sqrt{\left\{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2\right\}},$$

and
$$n = \sqrt{\left\{\left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2\right\}},$$

the direction-cosines of the tangents at P to PP_2 and PP_3 are
$$\frac{1}{m}\frac{\partial x}{\partial \mu}, \quad \frac{1}{m}\frac{\partial y}{\partial \mu}, \quad \frac{1}{m}\frac{\partial z}{\partial \mu},$$

and
$$\frac{1}{n}\frac{\partial x}{\partial \nu}, \quad \frac{1}{n}\frac{\partial y}{\partial \nu}, \quad \frac{1}{n}\frac{\partial z}{\partial \nu}.$$

It follows that the necessary and sufficient conditions that the system should be orthogonal are

$$\begin{aligned}
\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} &= 0, \\
\frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \nu} &= 0, \qquad (3) \\
\frac{\partial x}{\partial \nu}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \nu}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \nu}\frac{\partial z}{\partial \lambda} &= 0.
\end{aligned}$$

Again, let s_1, s_2, s_3 be the lengths of arcs of the curves PP_1, PP_2, PP_3 measured from fixed points on these arcs to P; then
$$\frac{ds_1}{d\lambda} = \sqrt{\left\{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2\right\}} = l,$$

and similarly (3′)
$$\frac{ds_2}{d\mu} = m, \quad \frac{ds_3}{d\nu} = n.$$

Hence, if the system is orthogonal, the elements of surface dS_1, dS_2, dS_3 of the surfaces P_2PP_3, P_3PP_1, P_1PP_2 at P are

$$\begin{aligned}
dS_1 &= ds_2 ds_3 = mn\,d\mu\,d\nu, \\
dS_2 &= ds_3 ds_1 = nl\,d\nu\,d\lambda, \qquad (3'') \\
dS_3 &= ds_1 ds_2 = lm\,d\lambda\,d\mu,
\end{aligned}$$

and the element of volume at P is
$$dV = ds_1 ds_2 ds_3 = lmn\,d\lambda\,d\mu\,d\nu. \quad . \quad . \quad (3''')$$

Now, if Green's Theorem (II., 6) with U = 1 be applied to the region bounded by the surfaces (Fig. 20) $PP_2P_4P_3$, $P_1P_5P_6P_7$, $PP_1P_7P_8$, $P_2P_5P_6P_4$, $PP_1P_5P_2$, $P_3P_7P_6P_4$ whose parameters are λ, $\lambda + \delta\lambda$, μ, $\mu + \delta\mu$, ν, $\nu + \delta\nu$, respectively, it gives

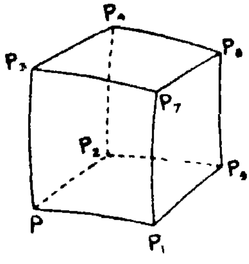

FIG. 20.

$$\begin{aligned}\iiint \nabla^2 V\, lmn\,d\lambda\,d\mu\,d\nu = \\ -\iint \frac{1}{l}\frac{\partial V}{\partial \lambda} mn\,d\mu\,d\nu + \iint \frac{1}{l}\frac{\partial V}{\partial \lambda} mn\,d\mu\,d\nu \\ -\iint \frac{1}{m}\frac{\partial V}{\partial \mu} nl\,d\nu\,d\lambda + \iint \frac{1}{m}\frac{\partial V}{\partial \mu} nl\,d\nu\,d\lambda \\ -\iint \frac{1}{n}\frac{\partial V}{\partial \nu} lm\,d\lambda\,d\mu + \iint \frac{1}{n}\frac{\partial V}{\partial \nu} lm\,d\lambda\,d\mu,\end{aligned} \tag{4}$$

where the integrals are taken over the six surfaces in the order given. In the first two of these integrals μ and ν have the same values, while the value of the remaining parameter is λ in the first and $\lambda + \delta\lambda$ in the second; hence the first line in the right hand side of equation (4) can be written

$$\iint \left[\frac{mn}{l}\frac{\partial V}{\partial \lambda}\right]_{\lambda}^{\lambda + \delta\lambda} d\mu\,d\nu = \iiint \frac{\partial}{\partial \lambda}\left(\frac{mn}{l}\frac{\partial V}{\partial \lambda}\right) d\lambda\,d\mu\,d\nu,$$

the volume integral being taken over the whole region. The remaining surface integrals can be transformed into volume integrals in the same manner, and we thus obtain the equation

$$\iiint \nabla^2 V\, lmn\,d\lambda\,d\mu\,d\nu =$$
$$\iiint \left\{\frac{\partial}{\partial \lambda}\left(\frac{mn}{l}\frac{\partial V}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{nl}{m}\frac{\partial V}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{lm}{n}\frac{\partial V}{\partial \nu}\right)\right\} lmn\,d\lambda\,d\mu\,d\nu.$$

Since this equation holds no matter how small the region of integration is taken, it follows that

$$\nabla^2 V = \frac{1}{lmn}\left\{\frac{\partial}{\partial \lambda}\left(\frac{mn}{l}\frac{\partial V}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{nl}{m}\frac{\partial V}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{lm}{n}\frac{\partial V}{\partial \nu}\right)\right\}, \tag{5}$$

and by putting the expression on the right equal to 0 or $-4\pi\rho$, we obtain the transformed equation of Laplace or Poisson as the case may be.

§ 2. Prolate Ellipsoids of Revolution. An ellipsoid of revolution, in which the axis of revolution is longer than the equal axes, is said to be *prolate* or elongated, while, if the axis of revolution is shorter than the equal axes, it is called *oblate* or flattened. In dealing with prolate ellipsoids of revolution a system of curvilinear co-ordinates (r, θ, ϕ) is employed which is given by the equations

$$x = \sqrt{(r^2 - e^2)} \sin\theta \cos\phi, \quad y = \sqrt{(r^2 - e^2)} \sin\theta \sin\phi, \quad z = r\cos\theta. \qquad (6)$$

When θ and ϕ are eliminated from (6), the equation

$$\frac{x^2 + y^2}{r^2 - e^2} + \frac{z^2}{r^2} = 1 \quad . \quad . \quad . \quad (7)$$

is obtained: this is a prolate ellipsoid of revolution, so that the r-system of surfaces consists of conicoids of that nature, all of which have the same foci $(0, 0, \pm e)$.

The elimination of r and ϕ leads to the equation

$$\frac{z^2}{e^2\cos^2\theta} - \frac{x^2 + y^2}{e^2\sin^2\theta} = 1, \quad . \quad . \quad (8)$$

which gives, as the θ-system of surfaces, a system of hyperboloids of rotation of two sheets, confocal with the system (7). Also, by elimination of r and θ, we find the ϕ-system to be the system of planes

$$y = x\tan\phi, \quad . \quad . \quad . \quad . \quad (9)$$

which pass through the axis of rotation, and consequently intersect all the surfaces (7) and (8) orthogonally. The surfaces of the systems (7) and (8) also intersect orthogonally: for the confocal conics in which they intersect the (x, z) plane are orthogonal, and the conicoids are obtained from these by rotation about the z-axis. The curvilinear system of co-ordinates (r, θ, ϕ) is therefore an orthogonal system, as can easily be verified by means of the formulæ (3).

To express the co-ordinates of all points uniquely in terms of r, θ, and ϕ, r is made to vary from e to ∞, θ from 0 to π, and ϕ from 0 to 2π. Then to each point of space there corresponds one and only one set of values (r, θ, ϕ), except that $\phi = 0$ and $\phi = 2\pi$ give the same points. When $r = e$, $x = y = 0$, $z = e\cos\theta$, which gives the points of the line joining the foci; this line may be regarded as a member of the system (7) whose equal axes have zero length. When $\theta = 0$, $x = y = 0$, $z = r$, and this gives the z-axis between $z = e$ and $z = +\infty$; $\theta = -\pi$ likewise gives the z-axis between $z = -e$ and $z = -\infty$. These lines form the two sheets of the member of the family (8) whose equal axes are of length zero. Similarly the angles θ and $\pi - \theta$ give, not distinct hyperboloids, but the two sheets of the same hyperboloid. Finally, when $\theta = \pi/2$, the hyperboloid becomes the (x, y) plane, both sheets coinciding in this plane.

Elliptic Co-ordinates. In (6) ϕ is the same angle as the corresponding angle in the system of polar co-ordinates: r is the semi-major axis of the ellipse in the plane $y = x \tan\phi$ whose foci are $(0, 0, \pm e)$ and which passes through the point (x, y, z); $\sqrt{(r^2 - e^2)}$ is the semi-minor axis of this ellipse, and θ is the eccentric angle of the point (x, y, z). For this reason these co-ordinates (r, θ, ϕ) are known as elliptic co-ordinates.

§ 3. **Laplace's Equation in Elliptic Co-ordinates.** From (2) we have, for the co-ordinates given in (6),

$$l = \frac{\sqrt{(r^2 - e^2\cos^2\theta)}}{\sqrt{(r^2 - e^2)}}, \quad m = \sqrt{(r^2 - e^2\cos^2\theta)}, \quad n = \sqrt{(r^2 - e^2)}\sin\theta, \tag{10}$$

and consequently, from (5),

$$\begin{aligned}\nabla^2 V &= \frac{1}{(r^2 - e^2\cos^2\theta)\sin\theta}\left[\frac{\partial}{\partial r}\left\{(r^2 - e^2)\sin\theta\frac{\partial V}{\partial r}\right\}\right. \\ &\qquad \left. + \frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial V}{\partial\theta}\right\} + \frac{\partial}{\partial\phi}\left\{\frac{r^2 - e^2\cos^2\theta}{(r^2 - e^2)\sin\theta}\frac{\partial V}{\partial\phi}\right\}\right] \\ &= \frac{1}{(r^2 - e^2\cos^2\theta)}\left[\frac{\partial}{\partial r}\left\{(r^2 - e^2)\frac{\partial V}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial V}{\partial\theta}\right\}\right. \\ &\qquad \left. + \left(\frac{e^2}{r^2 - e^2} + \frac{1}{\sin^2\theta}\right)\frac{\partial^2 V}{\partial\phi^2}\right]. \end{aligned} \tag{11}$$

Hence Laplace's Equation can be written

$$\frac{\partial}{\partial r}\left\{(r^2 - e^2)\frac{\partial V}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial V}{\partial\theta}\right\} + \left(\frac{1}{\sin^2\theta} + \frac{e^2}{r^2 - e^2}\right)\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{12}$$

To solve this equation put $V = V_1V_2V_3$, in (12), where V_1, V_2, and V_3 are functions respectively of r, θ, and ϕ alone, and divide by $V_1V_2V_3$; then

$$\frac{1}{V_1}\frac{d}{dr}\left\{(r^2 - e^2)\frac{dV_1}{dr}\right\} + \frac{1}{V_2\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dV_2}{d\theta}\right\} + \left(\frac{1}{\sin^2\theta} + \frac{e^2}{r^2 - e^2}\right)\frac{1}{V_3}\frac{d^2V_3}{d\phi^2} = 0. \quad (13)$$

Now the first and second terms in this equation are independent of ϕ, and therefore so is also the third; hence

$$\frac{1}{V_3}\frac{d^2V_3}{d\phi^2} = C, \quad . \quad . \quad . \quad (14)$$

where C is a constant. On solving this equation, we get

$$V_3 = A\cos\{\sqrt{(-C)}\cdot\phi\} + B\sin\{\sqrt{(-C)}\cdot\phi\},$$

where A and B are arbitrary constants.

Since V_3 is uniform, and has the same value for $\phi = 0$ and $\phi = 2\pi$, $\sqrt{(-C)}$ must be an integer; m say: thus (14) can be written

$$\frac{d^2V_3}{d\phi^2} + m^2V_3 = 0, \quad . \quad . \quad . \quad (15)$$

and

$$V_3 = A\cos m\phi + B\sin m\phi.$$

Again, from (15) and (13) it follows that

$$\frac{1}{V_1}\frac{d}{dr}\left\{(r^2 - e^2)\frac{dV_1}{dr}\right\} - \frac{m^2e^2}{r^2 - e^2} = -\frac{1}{V_2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV_2}{d\theta}\right) + \frac{m^2}{\sin^2\theta}.$$

Now the right-hand side of this equation is independent of r, and therefore so is also the left, the value of which must therefore be constant. It is convenient to put this constant in the form $\alpha(\alpha + 1)$; thus

$$\frac{d}{dr}\left\{(r^2 - e^2)\frac{dV_1}{dr}\right\} - \left\{\alpha(\alpha + 1) + \frac{m^2e^2}{r^2 - e^2}\right\}V_1 = 0, \quad (16)$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV_2}{d\theta}\right) + \left\{\alpha(\alpha + 1) - \frac{m^2}{\sin^2\theta}\right\}V_2 = 0. \quad (17)$$

In (17) put $\cos\theta = \mu$, and the equation becomes Legendre's Associated Equation—

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV_2}{d\mu}\right\} + \left\{\alpha(\alpha+1) - \frac{m^2}{1-\mu^2}\right\}V_2 = 0,$$

of which the general solution is

$$V_2 = AP_\alpha^m(\mu) + BQ_\alpha^m(\mu).$$

Now V_2 is finite when $\mu = 1$; but, from Ch. VI., *ex.* 9, and (VII., 4), it follows that $Q_\alpha^m(\mu)$ is not finite when $\mu = 1$: thus B must be zero. Again, from Ch. VI., *ex.* 10, and (VII., 3), we deduce that $P_\alpha^m(-1)$ is only finite when α is an integer. Hence α must have an integral value, n say, and therefore

$$V_2 = AP_n^m(\mu), \text{ or } V_2 = A'T_n^m(\mu),$$

where $m \leqq n$: if m were greater than n, V_2 would vanish identically.

Again, in (16) put $\alpha = n$, and write x for r/e; the equation is then transformed into Legendre's Associated Equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dV_1}{dx}\right\} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}V_1 = 0,$$

of which the general solution is

$$V_1 = AP_n^m\left(\frac{r}{e}\right) + BQ_n^m\left(\frac{r}{e}\right). \qquad (18)$$

If now a potential is required which remains finite for all points within the ellipsoid $r = a$, and, in particular, on the line $r = e$, B must be zero, since, as shown above, $Q_n^m(1)$ is infinite: hence

$$V_1 = AP_n^m\left(\frac{r}{e}\right).$$

Thus a solution of equation (12) is

$$V = P_n^m\left(\frac{r}{e}\right)T_n^m(\cos\theta)\{A\cos m\phi + B\sin m\phi\},$$

where A and B are arbitrary constants; the most general solution obtained in this way for the region interior to the ellipsoid $r = a$ is

$$V_i = \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m\left(\frac{r}{e}\right)T_n^m(\cos\theta)\{A_n^m \cos m\phi + B_n^m \sin m\phi\}, \quad (19)$$

where the constants A_n^m, B_n^m are arbitrary, provided only that they are chosen so that the series is convergent.

For the region external to the ellipsoid $r = a$ the constant A in (18) must vanish, since $P_n^m\left(\frac{r}{e}\right)$ is infinite when r is infinite. Accordingly, the most general solution obtained in this way for the region exterior to the ellipsoid $r = a$ is

$$V_e = \sum_{n=0}^{\infty}\sum_{m=0}^{n} Q_n^m\left(\frac{r}{e}\right)T_n^m(\cos\theta)\{C_n^m \cos m\phi + D_n^m \sin m\phi\}, \quad (20)$$

where the constants C_n^m and D_n^m are arbitrary.

Finally, the most general solution for the region bounded by the confocal ellipsoids of rotation $r = a$, $r = a_1$ is of the form

$$V = \sum_{n=0}^{\infty}\sum_{m=0}^{n} T_n^m(\cos\theta)\left[\begin{array}{c}\cos m\phi\left\{A_n^m P_n^m\left(\frac{r}{e}\right) + B_n^m Q_n^m\left(\frac{r}{e}\right)\right\} \\ + \sin m\phi\left\{C_n^m P_n^m\left(\frac{r}{e}\right) + D_n^m Q_n^m\left(\frac{r}{e}\right)\right\}\end{array}\right]. \quad (21)$$

Example 1. If the value of the potential V on the surface of the prolate ellipsoid of revolution $r = a$ is $V = F(\theta, \phi)$, where the function $F(\theta, \phi)$ can be expanded in the form

$$F(\theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} T_n^m(\cos\theta)\{A_n^m \cos m\phi + B_n^m \sin m\phi\},$$

and if V satisfies Laplace's Equation at all points within the ellipsoid, show that its value at these points is given by

$$V = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{P_n^m\left(\frac{r}{e}\right)}{P_n^m\left(\frac{a}{e}\right)} T_n^m(\cos\theta)\{A_n^m \cos m\phi + B_n^m \sin m\phi\}.$$

Example 2. Find the surface density of an electrical charge M on an isolated conductor bounded by the prolate ellipsoid of revolution $r = a$, and subject to no external forces.

The internal and external potentials V_i and V_e are given by (19) and (20) respectively. But V_i is constant, so that in (19) all the constants except that for which n and m are both zero must vanish: thus

$$V_i = A_0.$$

Again, V_e is constant on the surface of the conductor; *i.e.*, when $r = a$. Thus, in (20), all the constants vanish unless when n is zero; therefore

$$V_e = C_0 Q_0\left(\frac{r}{e}\right).$$

But, on the surface $r = a$, $V_e = V_i$: hence

$$C_0 Q_0\left(\frac{a}{e}\right) = A_0,$$

and therefore, at any external point,

$$V_e = A_0 \frac{Q_0\left(\frac{r}{e}\right)}{Q_0\left(\frac{a}{e}\right)}.$$

Now let σ be the surface density at the point (a, θ, ϕ); then

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V_e}{\partial N} - \frac{\partial V_i}{\partial N}\right) = -\frac{A_0}{4\pi}\frac{1}{Q_0\left(\frac{a}{e}\right)}\frac{d}{dN}Q_0\left(\frac{r}{e}\right),$$

where N denotes a distance along the normal measured outwards from the surface. But, by (3′),

$$\frac{dN}{dr} = l, \text{ where } l = \sqrt{\left(\frac{r^2 - e^2\cos^2\theta}{r^2 - e^2}\right)},$$

by (10), and the positive sign of the square root is taken since N increases with r; hence

$$\sigma = -\frac{A_0}{4\pi}\frac{1}{Q_0\left(\frac{a}{e}\right)}\sqrt{\left(\frac{a^2 - e^2}{a^2 - e^2\cos^2\theta}\right)}\left[\frac{d}{dr}Q_0\left(\frac{r}{e}\right)\right]_{r=a};$$

and, since (Ch. VI. § 1, ex.),

$$Q_0\left(\frac{r}{e}\right) = \frac{1}{2}\log\left(\frac{r+e}{r-e}\right),$$

$$\frac{d}{dr}Q_0\left(\frac{r}{e}\right) = -\frac{e}{r^2 - e^2},$$

and therefore

$$\sigma = \frac{A_0 e}{4\pi Q_0\left(\frac{a}{e}\right)\sqrt{(a^2 - e^2)}}\,\frac{1}{\sqrt{(a^2 - e^2\cos^2\theta)}}.$$

Now the total charge on the conductor is

$$M = \iint \sigma dS,$$

the integral being taken over the surface of the ellipsoid. But, by (3″) and (10), this can be written

$$M = \iint \sigma m n d\theta d\phi = \frac{A_0 e}{4\pi Q_0\left(\frac{a}{e}\right)} \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta d\phi$$

$$= \frac{A_0 e}{Q_0\left(\frac{a}{e}\right)}.$$

Hence, finally,

$$\sigma = \frac{M}{4\pi\sqrt{(a^2 - e^2)}} \frac{1}{\sqrt{(a^2 - e^2\cos^2\theta)}},$$

and

$$V_i = \frac{M}{e} Q_0\left(\frac{a}{e}\right), \quad V_e = \frac{M}{e} Q_0\left(\frac{r}{e}\right).$$

Example 3. Show that the density at any point on the surface of the ellipsoid of *ex.* 2 is proportional to the length of the perpendicular from the centre to the tangent plane at the point.

§ 4. Expression for the Reciprocal of the Distance between two Points in Elliptic Co-ordinates.

Let R denote the distance between the points P, P_1, whose rectangular co-ordinates are (x, y, z), (x_1, y_1, z_1), and whose elliptic co-ordinates are (r, θ, ϕ), (r_1, θ_1, ϕ_1) respectively; then, from (6),

$$x = \sqrt{(r^2 - e^2)} \sin\theta \cos\phi, \qquad x_1 = \sqrt{(r_1^2 - e^2)} \sin\theta_1 \cos\phi_1,$$
$$y = \sqrt{(r^2 - e^2)} \sin\theta \sin\phi, \qquad y_1 = \sqrt{(r_1^2 - e^2)} \sin\theta_1 \sin\phi_1,$$
$$z = r\cos\theta, \qquad z_1 = r_1 \cos\theta_1,$$

and

$$\begin{aligned} R^2 &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \\ &= r^2 + r_1^2 - e^2(\sin^2\theta + \sin^2\theta_1) \\ &\qquad - 2\sqrt{(r^2 - e^2)}\sqrt{(r_1^2 - e^2)} \sin\theta \sin\theta_1 \cos(\phi - \phi_1) \\ &\qquad - 2rr_1 \cos\theta \cos\theta_1 \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

Now, assume that $r > r_1$; or, that there is a value r_2 of r such that $r_1 < r_2 < r$; so that P lies outside and P_1 inside the ellipsoid $r = r_2$. Then, since $1/R$ satisfies Laplace's Equation, it can be expressed in the form (20)

$$\frac{1}{R} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} Q_n^m\left(\frac{r}{e}\right) T_n^m(\cos\theta)\{C_n^m \cos m\phi + D_n^m \sin m\phi\}. \quad (23)$$

For, on the surface $r = r_2$, $1/R$ is a function of θ and ϕ, and may therefore, by (VII., 41, 42), be expressed in the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} T_n^m(\cos\theta)\{A_n^m \cos m\phi + B_n^m \sin m\phi\}.$$

But, if the constants C_n^m, D_n^m are chosen so that

$$Q_n^m\left(\frac{r_2}{e}\right)C_n^m = A_n^m, \quad Q_n^m\left(\frac{r_2}{e}\right)D_n^m = B_n^m,$$

the series on the right of (23) and $1/R$ are equal on $r = r_2$; hence, as both these functions tend to zero when r tends to infinity, they are equal (Ch. IX., § 1, Theorem 1) for $r \geqq r_2$.

Now, from (22), it is clear that in (23) ϕ only occurs in the combination $\phi - \phi_1$, and that $1/R$ is an even function of this quantity; hence

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} E_n^m Q_n^m\left(\frac{r}{e}\right)T_n^m(\cos\theta)\cos m(\phi - \phi_1), \qquad (23')$$

where E_n^m is independent of r, θ, ϕ, and ϕ_1, but may be a function of r_1 and θ_1.

Again, if $1/R$ is regarded as a function of r_1, θ_1, ϕ_1, it can be expressed in the form (19)

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} K_n^m P_n^m\left(\frac{r_1}{c}\right)T_n^m(\cos\theta_1)\cos m(\phi - \phi_1), \qquad (24)$$

where K_n^m is independent of r_1, θ_1, ϕ_1, and ϕ, but may be a function of r and θ. Also, if θ and θ_1 are interchanged, r, r_1, ϕ, and ϕ_1 being kept constant, the value of R is unaltered. Hence, from (23'), if $E_n^m = f(r_1, \theta_1)$

$$f(r_1, \theta_1)T_n^m(\cos\theta) = f(r_1, \theta)T_n^m(\cos\theta_1),$$

so that

$$\frac{f(r_1, \theta_1)}{T_n^m(\cos\theta_1)} = \frac{f(r_1, \theta)}{T_n^m(\cos\theta)}.$$

Now the right-hand side of this equation is independent of θ_1, and consequently the left-hand side is also independent of θ_1; hence

$$E_n^m = T_n^m(\cos\theta_1)F_n^m,$$

where F_n^m is a function of r_1 alone. Similarly,

$$K_n^m = T_n^m(\cos\theta)L_n^m,$$

where L_n^m is a function of r alone. Thus (23′) and (24) become

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} F_n^m Q_n^m\left(\frac{r}{e}\right)T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi-\phi_1), \quad (25)$$

and

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} L_n^m P_n^m\left(\frac{r_1}{e}\right)T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi-\phi_1). \quad (26)$$

Equations (25), (26) give, for any values of r and r_1, expressions for $1/R$ in terms of series of harmonics of θ and ϕ. But, since these must be equal for all values of θ and ϕ, the spherical harmonics of equal orders must be identical, and the coefficients of each cosine must be equal ; thus

$$F_n^m Q_n^m\left(\frac{r}{e}\right) = L_n^m P_n^m\left(\frac{r_1}{e}\right),$$

or

$$\frac{F_n^m}{P_n^m\left(\frac{r_1}{e}\right)} = \frac{L_n^m}{Q_n^m\left(\frac{r}{e}\right)}.$$

Now the left-hand side of this equation is independent of r, and therefore so is also the right-hand side , similarly each side is independent of r_1. Thus each fraction is equal to a constant H_n^m, and

$$F_n^m Q_n^m\left(\frac{r}{e}\right) = L_n^m P_n^m\left(\frac{r_1}{e}\right) = H_n^m Q_n^m\left(\frac{r}{e}\right)P_n^m\left(\frac{r_1}{e}\right).$$

Accordingly

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} H_n^m Q_n^m\left(\frac{r}{e}\right)P_n^m\left(\frac{r_1}{e}\right)T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi-\phi_1).$$

The function H_n^m may be a function of e. Let e tend to zero, and the ellipsoids become spheres, while the co-ordinates become ordinary polar co-ordinates, and

$$\frac{1}{\mathrm{R}} = \frac{1}{\sqrt{(r^2 + r_1^2 - 2rr_1 \cos\gamma)}} = \sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} \mathrm{P}_n(\cos\gamma)$$

$$= \sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} \Big\{ \mathrm{P}_n(\cos\theta)\mathrm{P}_n(\cos\theta_1)$$

$$+ 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \mathrm{T}_n^m(\cos\theta)\mathrm{T}_n^m(\cos\theta_1) \ \cos m(\phi - \phi_1) \Big\},$$

by (VII., 34): thus

$$\operatorname*{Lim}_{e \to 0} \left\{ \mathrm{H}_n^m \mathrm{Q}_n^m\left(\frac{r}{e}\right) \mathrm{P}_n^m\left(\frac{r_1}{e}\right) \right\} = 2 \frac{r_1^n}{r^{n+1}} \frac{(n-m)!}{(n+m)!},$$

or

$$\operatorname*{Lim}_{e \to 0} \left\{ \mathrm{H}_n^m r^{n+1} \mathrm{Q}_n^m\left(\frac{r}{e}\right) r_1^{-n} \mathrm{P}_n^m\left(\frac{r_1}{e}\right) \right\} = 2 \frac{(n-m)!}{(n+m)!},$$

the 2 being omitted when m is zero. But, from (V., 11) and (VII., 3)

$$\operatorname*{Lim}_{e \to 0} \left(\frac{e}{r_1}\right)^n \mathrm{P}_n^m\left(\frac{r_1}{e}\right) = \frac{(2n)!}{2^n n!\,(n-m)!},$$

and, from (VI., 6) and (VII., 4),

$$\operatorname*{Lim}_{e \to 0} \left(\frac{r}{e}\right)^{n+1} \mathrm{Q}_n^m\left(\frac{r}{e}\right) = \frac{\mathrm{B}(\frac{1}{2}, n+1)}{2^{n+1}} (-1)^m \frac{(n+m)!}{n!}.$$

Hence

$$\operatorname*{Lim}_{e \to 0} \mathrm{H}_n^m e \left(\frac{r}{e}\right)^{n+1} \mathrm{Q}_n^m\left(\frac{r}{e}\right) \left(\frac{e}{r_1}\right)^n \mathrm{P}_n^m\left(\frac{r_1}{e}\right)$$

$$= \operatorname*{Lim}_{e \to 0} (\mathrm{H}_n^m e)(-1)^m \frac{(n+m)!}{(n-m)!} \frac{(2n)!}{2^{2n+1}(n!)^2} \mathrm{B}(\tfrac{1}{2}, n+1)$$

$$= \operatorname*{Lim}_{e \to 0} (\mathrm{H}_n^m e)(-1)^m \frac{(n+m)!}{(n-m)!} \frac{1}{2n+1},$$

and therefore

$$\operatorname*{Lim}_{e \to 0} (\mathrm{H}_n^m e) = (-1)^m 2(2n+1) \left\{ \frac{(n-m)!}{(n+m)!} \right\}^2,$$

the 2 being omitted when m is zero. Denote this constant by X_n^m, and it follows that

$$\mathrm{H}_n^m = \frac{1}{e} \mathrm{X}_n^m f_n^m(e),$$

where $f_n^m(e)$ is a function of e which has the value 1 when e is zero. The formula for $\frac{1}{\mathrm{R}}$ can now be written

$$\frac{e}{\mathrm{R}}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathrm{X}_n^m f_n^m(e)\mathrm{Q}_n^m\left(\frac{r}{e}\right)\mathrm{P}_n^m\left(\frac{r_1}{e}\right)\mathrm{T}_n^m(\cos\theta)\mathrm{T}_n^m(\cos\theta_1)\cos m(\phi-\phi_1) \tag{27}$$

But, from (22),

$$\left(\frac{\mathrm{R}}{e}\right)^2=\left(\frac{r}{e}\right)^2+\left(\frac{r_1}{e}\right)^2-\sin^2\theta-\sin^2\theta_1-2\sqrt{\left\{\left(\frac{r}{e}\right)^2-1\right\}}$$
$$\times\sqrt{\left\{\left(\frac{r_1}{e}\right)^2-1\right\}}\sin\theta\sin\theta_1\cos(\phi-\phi_1)-2\frac{r}{e}\frac{r_1}{e}\cos\theta\cos\theta_1,$$

from which it is clear that e/R only contains e in the combinations r/e and r_1/e, and this must be true of the expression in (27). The functions $f_n^m(e)$ must therefore be independent of e, so that, for all values of e, it has the same value unity. Thus, finally, we have the result

$$\frac{1}{\mathrm{R}}=\frac{1}{e}\sum_{n=0}^{\infty}(2n+1)\left[\begin{array}{l}\mathrm{Q}_n\left(\frac{r}{e}\right)\mathrm{P}_n\left(\frac{r_1}{e}\right)\mathrm{P}_n(\cos\theta)\mathrm{P}_n(\cos\theta_1)\\ \quad+2\sum^{n}(-1)^m\left\{\frac{(n-m)!}{(n+m)!}\right\}^2\mathrm{Q}_n^m\left(\frac{r}{e}\right)\mathrm{P}_n^m\left(\frac{r_1}{e}\right)\\ \quad\times\mathrm{T}_n^m(\cos\theta)\mathrm{T}_n^m(\cos\theta_1)\cos m(\phi-\phi_1)\end{array}\right] \tag{28}$$

Example 1. Show that

$$\frac{e}{r-e\cos\theta}=\sum_{n=0}^{\infty}(2n+1)\mathrm{Q}_n\left(\frac{r}{e}\right)\mathrm{P}_n(\cos\theta).$$

[Put $r_1=e$, $\theta_1=0$.]

Example 2. From the previous example deduce that, if $y>1$, $x^2\leqq 1$,

$$\frac{1}{y-x}=\sum_{n=0}^{\infty}(2n+1)\mathrm{Q}_n(y)\mathrm{P}_n(x).$$

Example 3. Show that

$$\frac{e}{\sqrt{(r^2-e^2\sin^2\theta)}}=\sum_{n=0}^{\infty}(4n+1)\mathrm{P}_{2n}(0)\mathrm{Q}_{2n}\left(\frac{r}{e}\right)\mathrm{P}_{2n}(\cos\theta).$$

[Put $r_1=e$, $\theta_1=\frac{1}{2}\pi$.]

Example 4. Find in elliptic co-ordinates the potential at an external point of the attraction of a prolate ellipsoid of revolution of uniform density σ bounded by the surface $r = a$.

Let V be the required potential at the point (r, θ, ϕ); then, from (3‴) and (10)

$$V = \sigma\int_e^a\int_0^\pi\int_0^{2\pi}\frac{(r_1^2 - e^2\cos^2\theta_1)\sin\theta_1\,d\phi_1 d\theta_1 dr_1}{R}.$$

If the series in (28) *be substituted for* $1/R$ *in this integral, and each term integrated with regard to* ϕ, *it is evident that the integrals of the terms* in which m is not zero will all vanish; thus

$$V = \frac{2\pi\sigma}{e}\sum_{n=0}^{\infty}(2n + 1)Q_n\left(\frac{r}{e}\right)P_n(\cos\theta)$$

$$\times\int_e^a\int_0^\pi P_n\left(\frac{r_1}{e}\right)P_n(\cos\theta_1)(r_1^2 - e^2\cos^2\theta_1)\sin\theta_1 d\theta_1 dr_1.$$

Now, if $\mu_1 = \cos\theta_1$,

$$r_1^2 - e^2\cos^2\theta_1 = (r_1^2 - \tfrac{1}{3}e^2)P_0(\mu_1) - \tfrac{2}{3}e^2P_2(\mu_1),$$

so that

$$V = \frac{2\pi\sigma}{e}\left\{\begin{aligned}&Q_0\left(\frac{r}{e}\right)P_0(\cos\theta)\int_e^a P_0\left(\frac{r_1}{e}\right)\left(r_1^2 - \frac{1}{3}e^2\right)2dr_1\\ &\quad - 5Q_2\left(\frac{r}{e}\right)P_2(\cos\theta)\int_e^a P_2\left(\frac{r_1}{e}\right)\frac{2}{3}e^2\frac{2}{5}dr_1\end{aligned}\right\}$$

$$= \frac{4\pi\sigma}{3e}a(a^2 - e^2)\left\{Q_0\left(\frac{r}{e}\right) - Q_2\left(\frac{r}{e}\right)P_2(\cos\theta)\right\}.$$

If for these Legendre Functions their values as given in Ch. VI., § 1, *ex.*, § 4, *ex.* 1 are taken, this formula can be written

$$V = \frac{\pi\sigma a(a^2 - e^2)}{e}\left[\frac{1}{2}\log\left(\frac{r+e}{r-e}\right)\left\{\frac{4}{3} - \frac{3r^2 - e^2}{3e^2}(3\cos^2\theta - 1)\right\} + \frac{r}{e}(3\cos^2\theta - 1)\right]$$

$$= \frac{\pi\sigma a(a^2 - e^2)}{e}\left[\frac{1}{2}\log\left(\frac{r+e}{r-e}\right)\left\{2 - \frac{2r^2\cos^2\theta}{e^2} + \frac{(r^2 - e^2)\sin^2\theta}{e^2}\right\}\right.$$

$$\left. + \frac{r(2\cos^2\theta - \sin^2\theta)}{e}\right].$$

Example 5. For the ellipsoid of *ex.* 4, find the potential at an internal point.

In this case the integral with regard to r is divided into two parts with limits e and r, and r and a respectively. In the case of the second of these integrals r and r_1 are interchanged in the expression (28) for $\frac{1}{R}$; thus

$$V = \frac{4\pi\sigma}{e}\left\{\begin{aligned}&Q_0\left(\frac{r}{e}\right)\int_e^r\left(r_1^2 - \frac{1}{3}e^2\right)dr_1 - \frac{2}{3}e^2Q_2\left(\frac{r}{e}\right)P_2(\cos\theta)\int_e^r P_2\left(\frac{r_1}{e}\right)dr_1\\ &+ \int_r^a\left(r_1^2 - \frac{1}{3}e^2\right)Q_0\left(\frac{r_1}{e}\right)dr_1 - \frac{2}{3}e^2P_2\left(\frac{r}{e}\right)P_2(\cos\theta)\int_r^a Q_2\left(\frac{r_1}{e}\right)dr_1\end{aligned}\right\}.$$

On integrating it is found that

$$V = \frac{4\pi\sigma}{3e} a(a^2 - e^2)Q_0\left(\frac{a}{e}\right)\left\{1 - P_2\left(\frac{r}{e}\right)P_2(\cos\theta)\right\}$$
$$+ 2\pi\sigma\left\{\frac{a^2 - r^2}{3} + P_2(\cos\theta)\left(\frac{a^2r^2}{e^2} - \frac{1}{3}a^2 - \frac{2}{3}r^2\right)\right\}.$$

Example 6. A point-charge m of electricity is placed at a point on the z-axis at a distance r_1 from the origin, and is situated outside an isolated conductor bounded by the prolate ellipsoid of revolution $r = a$, where $a < r_1$: find the surface density of the distribution of electricity on the conductor.

The potential of the point-charge at the point (r, θ, ϕ) is

$$U = \frac{m}{R},$$

where, by (28), since $r_1 > a > r$ for points within the ellipsoid $r = a$, and $\theta_1 = 0$,

$$\frac{1}{R} = \frac{1}{e}\sum_{n=0}^{\infty}(2n + 1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{r}{e}\right)P_n(\cos\theta).$$

Now let V_i be the potential due to the surface distribution at an internal point; then

$$V_i + U = C,$$

where C is a constant; so that

$$V_i = C - \frac{m}{e}\sum_{n=0}^{\infty}(2n + 1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{r}{e}\right)P_n(\cos\theta);$$

and, in particular, at a point on the surface $r = a$,

$$(V_i)_{r=a} = C - \frac{m}{e}\sum_{n=0}^{\infty}(2n + 1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{a}{e}\right)P_n(\cos\theta).$$

Again, if V_e be the potential of the surface distribution at an external point, then, by (20),

$$V_e = \sum_{n=0}^{\infty}C_nQ_n\left(\frac{r}{e}\right)P_n(\cos\theta),$$

the terms in which m is not zero being omitted, since, from the symmetry of the distribution, V_e is independent of ϕ : in particular, on the surface $r = a$,

$$(V_e)_{r=a} = \sum_{n=0}^{\infty}C_nQ_n\left(\frac{a}{e}\right)P_n(\cos\theta).$$

But, at all points of the surface $r = a$,

$$(V_i)_{r=a} = (V_e)_{r=a},$$

and therefore the coefficients of the Legendre Coefficients in this equation are equal; thus

$$C_n Q_n\left(\frac{a}{e}\right) = -\frac{m}{e}(2n+1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{a}{e}\right),\ n = 1, 2, 3, \ldots$$

and

$$C_0 Q_0\left(\frac{a}{e}\right) = C - \frac{m}{e}Q_0\left(\frac{r_1}{e}\right)P_0\left(\frac{a}{e}\right).$$

It follows that

$$V_e = C\frac{Q_0\left(\frac{r}{e}\right)}{Q_0\left(\frac{a}{e}\right)} - \frac{m}{e}\sum_{n=0}^{\infty}(2n+1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{a}{e}\right)\frac{Q_n\left(\frac{r}{e}\right)}{Q_n\left(\frac{a}{e}\right)}P_n(\cos\theta).$$

Now, let σ be the density at the point (a, θ, ϕ) on the surface of the conductor; then

$$\sigma = -\frac{1}{4\pi}\left\{\frac{\partial V_e}{\partial N} - \frac{\partial V_i}{\partial N}\right\} = -\frac{1}{4\pi}\left\{\frac{\partial V_e}{\partial r} - \frac{\partial V_i}{\partial r}\right\}\frac{dr}{dN},$$

where N denotes a distance along the normal measured outwards. Thus, as in § 3, ex. 2,

$$\sigma = -\frac{1}{4\pi}\sqrt{\left(\frac{a^2 - e^2}{a^2 - e^2\cos^2\theta}\right)}$$

$$\times\left\{\begin{array}{l}\frac{C}{e}\frac{Q_0'\left(\frac{a}{e}\right)}{Q_0\left(\frac{a}{e}\right)} - \frac{m}{e^2}\sum_{n=0}^{\infty}(2n+1)Q_n\left(\frac{r_1}{e}\right)P_n\left(\frac{a}{e}\right)\frac{Q'_n\left(\frac{a}{e}\right)}{Q_n\left(\frac{a}{e}\right)}P_n(\cos\theta)\\ \qquad\qquad + \frac{m}{e^2}\sum_{n=0}^{\infty}(2n+1)Q_n\left(\frac{r_1}{e}\right)P'_n\left(\frac{a}{e}\right)P_n(\cos\theta)\end{array}\right\}.$$

But (Ch. VI., § 5, Cor.),

$$Q_n\left(\frac{a}{e}\right)P'_n\left(\frac{a}{e}\right) - Q'_n\left(\frac{a}{e}\right)P_n\left(\frac{a}{e}\right) = \frac{e^2}{a^2 - e^2},$$

and therefore

$$\sigma = -\frac{1}{4\pi}\sqrt{\left(\frac{a^2 - e^2}{a^2 - e^2\cos^2\theta}\right)}\left\{\frac{C}{e}\frac{Q'_0\left(\frac{a}{e}\right)}{Q_0\left(\frac{a}{e}\right)}\right.$$

$$\left. + \frac{m}{a^2 - e^2}\sum_{n=0}^{\infty}(2n+1)\frac{Q_n\left(\frac{r_1}{e}\right)}{Q_n\left(\frac{a}{e}\right)}P_n(\cos\theta)\right\}.$$

If M is the total charge on the conductor, it can easily be verified that

$$\mathrm{M} = -\frac{\mathrm{C}}{e}(a^2 - e^2)\frac{\mathrm{Q}_0'\left(\frac{a}{e}\right)}{\mathrm{Q}_0\left(\frac{a}{e}\right)} - m\frac{\mathrm{Q}_0\left(\frac{r_1}{e}\right)}{\mathrm{Q}_0\left(\frac{a}{e}\right)}.$$

§ 5. **Oblate Ellipsoids of Revolution.** The system of co-ordinates employed in connection with oblate or flattened ellipsoids of revolution is given by the equation

$$x = \sqrt{(r^2 + e^2)}\sin\theta\cos\phi,$$
$$y = \sqrt{(r^2 + e^2)}\sin\theta\sin\phi, \quad z = r\cos\theta. \qquad (29)$$

The elimination of θ and ϕ from these equations leads to the equation

$$\frac{x^2 + y^2}{r^2 + e^2} + \frac{z^2}{r^2} = 1, \quad . \quad . \quad . \quad (30)$$

which represents an oblate ellipsoid of revolution, obtained by revolving the ellipse $\frac{x^2}{r^2 + e^2} + \frac{z^2}{r^2} = 1$, $y = 0$ about the z-axis: these ellipses have all the same foci ($\pm e$, 0, 0), and, on revolution, these foci trace out the circle

$$x^2 + y^2 = e^2, \quad z = 0.$$

On eliminating r and ϕ from (29), we obtain the equation

$$\frac{x^2 + y^2}{e^2\sin^2\theta} - \frac{z^2}{e^2\cos^2\theta} = 1 \quad . \quad . \quad . \quad (31)$$

which is the equation of the hyperboloid of one sheet obtained by revolving the hyperbola

$$\frac{x^2}{e^2\sin^2\theta} - \frac{z^2}{e^2\cos^2\theta} = 1, \; y = 0$$

about the z-axis: this hyperbola is confocal with the ellipse mentioned above, so that these curves, and consequently the ellipsoid (30) and the hyperboloid (31), intersect orthogonally.

If r and θ are eliminated from (29), the resulting equation is

$$y = x\tan\phi,$$

which represents a system of planes through the z-axis, cutting the ellipsoids (30) and the hyperboloids (31) orthogonally.

The system of co-ordinates (r, θ, ϕ) forms a special case of the elliptic co-ordinates, and can be obtained from those of § 2 by replacing e^2 by $-e^2$.

The co-ordinates of all points of space are obtained by making r vary from 0 to ∞, θ from 0 to π, and ϕ from 0 to 2π: then, to each point of space, except those mentioned below, there corresponds a unique set of co-ordinates of the system.

When $r = 0$, the corresponding ellipsoid has no thickness, and reduces to that part of the (x, y) plane which is bounded by the circle traced out by the foci; the equations (29) then become

$$x = e \sin\theta \cos\phi, \quad y = e \sin\theta \sin\phi, \quad z = 0,$$

and two values of θ, namely, θ and $\pi - \theta$, give the cartesian co-ordinates of the same point; the values of θ from 0 to $\frac{1}{2}\pi$ are regarded as corresponding to the upper surface of this flattened ellipsoid, those from $\frac{1}{2}\pi$ to π as belonging to the lower surface.

For $\theta = 0$ and $\theta = \pi$ the hyperboloid (31) degenerates into the positive and negative parts of the z-axis respectively: for each point on the z-axis ϕ may have any value. When $\theta = \frac{1}{2}\pi$ the hyperboloid reduces to the part of the (x, y) plane external to the focal circle. In general, the values θ and $\pi - \theta$ of θ give points on the same hyperboloid, values of θ from 0 to $\frac{1}{2}\pi$ giving points for which z is positive, and values of θ from $\frac{1}{2}\pi$ to π giving points for which z is negative.

§ 6. **Transformation and Solution of Laplace's Equation.** For the co-ordinates discussed in the previous section, formulæ (2) give

$$l = \frac{\sqrt{(r^2 + e^2\cos^2\theta)}}{\sqrt{(r^2 + e^2)}}, \quad m = \sqrt{(r^2 + e^2\cos^2\theta)},$$
$$n = \sqrt{(r^2 + e^2)}\sin\theta, \qquad (32)$$

and, from (5), it follows that

$$\nabla^2 V = \frac{1}{r^2 + e^2\cos^2\theta}\left[\frac{\partial}{\partial r}\left\{(r^2 + e^2)\frac{\partial V}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial V}{\partial\theta}\right\} + \left(\frac{1}{\sin^2\theta} - \frac{e^2}{r^2 + e^2}\right)\frac{\partial^2 V}{\partial\phi^2}\right], \qquad (33)$$

so that Laplace's Equation becomes

$$\frac{\partial}{\partial r}\left\{(r^2 + e^2)\frac{\partial V}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial V}{\partial\theta}\right\} + \left(\frac{1}{\sin^2\theta} - \frac{e^2}{r^2 + e^2}\right)\frac{\partial^2 V}{\partial\phi^2} = 0. \quad (34)$$

As in § 3, let $V = V_1V_2V_3$, where V_1, V_2, V_3 are functions of r, θ, ϕ alone; then, proceeding as before, we find that

$$V_3 = A\cos m\phi + B\sin m\phi,$$

and

$$V_2 = CT_n^m(\cos\theta),$$

where m and n are integers and $n \geqq m$; while the differential equation for V_1 is

$$\frac{d}{dr}\left\{(r^2 + e^2)\frac{dV_1}{dr}\right\} - \left\{n(n+1) - \frac{m^2e^2}{r^2 + e^2}\right\}V_1 = 0. \quad (35)$$

In this equation put $x = \frac{ir}{e}$, and the equation reduces to Legendre's Associated Equation

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dV_1}{dx}\right\} + \left\{n(n+1) - \frac{m^2}{1 - x^2}\right\}V_1 = 0.$$

The general solution of (35) is therefore

$$V_1 = AP_n^m\left(\frac{ir}{e}\right) + BQ_n^m\left(\frac{ir}{e}\right).$$

From formulæ (V., 6) and (VII., 3) it is clear that $P_n^m\left(\frac{ir}{e}\right)$ is either real or purely imaginary, so that, by a proper choice of the constant A, it is always possible to make $AP_n^m\left(\frac{ir}{e}\right)$ real; similarly it follows from (VII., 7) that B can always be chosen so that $BQ_n^m\left(\frac{ir}{e}\right)$ is real.

For the region exterior to the ellipsoid $r = a$, A must be zero, since $P_n^m\left(\frac{ir}{e}\right)$, being the product of $\left(\frac{r^2}{e^2} + 1\right)^{\frac{1}{2}m}$ and a polynomial in $\frac{r}{e}$, tends to infinity when r tends to infinity; on the other hand, if $m \leqq n$, $Q_n^m\left(\frac{ir}{e}\right)$ tends to zero when r tends to infinity. Thus the most general solution of this kind for the region outside $r = a$ is of the form

$$V_e = \sum_{n=0}^{\infty} \sum_{m=0}^{n} Q_n^m\left(\frac{ir}{e}\right) T_n^m(\cos\theta)\left\{A_n^m \cos m\phi + B_n^m \sin m\phi\right\}, \quad (36)$$

where the constants are arbitrary, apart from the condition that the series must be convergent.

For the space between the ellipsoids $r = a$, $r = a_1$, the general solution is

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_n^m(\cos\theta)\left[\begin{array}{l} \cos m\phi\left\{A_n^m P_n^m\left(\frac{ir}{e}\right) + B_n^m Q_n^m\left(\frac{ir}{e}\right)\right\} \\ + \sin m\phi\left\{C_n^m P_n^m\left(\frac{ir}{e}\right) + D_n^m Q_n^m\left(\frac{ir}{e}\right)\right\} \end{array}\right]. \quad (37)$$

For the space interior to the ellipsoid $r = a$, it is not the case that $Q_n^m\left(\frac{ir}{e}\right)$ is infinite when $r = e$, since $Q_n^m(i)$ is finite (cf. *ex.* 9, p. 118); it will, however, be shown by a different method that, in order that the solution

$$V = T_n^m(\cos\theta) \cos m\phi\left\{AP_n^m\left(\frac{ir}{e}\right) + BQ_n^m\left(\frac{ir}{e}\right)\right\}$$

should be valid for this region, B must be zero. For the solution to be valid, it is necessary not merely that V, but also that the derivatives of V should be finite throughout the region. The same proof applies when $\cos m\phi$ is replaced by $\sin m\phi$.

Let N and N_1 denote distances along the normals to the ellipsoid and hyperboloid through the point (r, θ, ϕ), measured in the directions of r and θ increasing respectively. Then

$$\frac{\partial V}{\partial N} = \frac{1}{l}\frac{\partial V}{\partial r} = \frac{\sqrt{(r^2+e^2)}}{\sqrt{(r^2+e^2\cos^2\theta)}} T_n^m(\cos\theta) \cos m\phi$$
$$\times\left\{A\frac{d}{dr}P_n^m\left(\frac{ir}{e}\right) + B\frac{d}{dr}Q_n^m\left(\frac{ir}{e}\right)\right\}, \quad (38)$$

and

$$\frac{\partial V}{\partial N_1} = \frac{1}{m}\frac{\partial V}{\partial \theta} = \frac{1}{\sqrt{(r^2+e^2\cos^2\theta)}} \frac{d}{d\theta}T_n^m(\cos\theta) \cos m\phi$$
$$\times\left\{AP_n^m\left(\frac{ir}{e}\right) + BQ_n^m\left(\frac{ir}{e}\right)\right\}. \quad (39)$$

Now, consider the points for which $\theta = \frac{1}{2}\pi$; *i.e.*, the points on the (x, y) plane. If $n - m$ is odd, $T_n^m(\cos\theta)$ has the value zero, since $\cos\theta = 0$, while $\frac{d}{d\theta}T_n^m(\cos\theta)$ is not zero. On the other hand, if $n - m$ is even, $T_n^m(\cos\theta)$ is not zero, while $\frac{d}{d\theta}T_n^m(\cos\theta)$ vanishes. Thus, when $\theta = \frac{1}{2}\pi$, if $n - m$ is odd, (38) vanishes; for $n - m$ even, (39) vanishes: on the other hand, when $\theta = \frac{1}{2}\pi$, and $n - m$ is even, (38) does not vanish, but becomes

$$\frac{\partial V}{\partial N} = \frac{\sqrt{(r^2 + e^2)}}{r} T_n^m(0) \cos m\phi \left\{ A \frac{d}{dr} P_n^m\left(\frac{ir}{e}\right) + B \frac{d}{dr} Q_n^m\left(\frac{ir}{e}\right) \right\}, \tag{38'}$$

while, when $n - m$ is odd, (39) does not vanish, but becomes

$$\frac{\partial V}{\partial N_1} = \frac{1}{r}\left[\frac{d}{d\theta} T_n^m(\cos\theta)\right]_{\theta = \frac{1}{2}\pi} \cos m\phi \left\{ A P_n^m\left(\frac{ir}{e}\right) + B Q_n^m\left(\frac{ir}{e}\right) \right\}. \tag{39'}$$

Now (Ch. VII., *ex.* 9), $Q_n^m\left(\frac{ir}{e}\right)$ and $\frac{d}{dr}Q_n^m\left(\frac{ir}{e}\right)$ do not vanish when $r = 0$: on the other hand, $\frac{d}{dr}P_n^m\left(\frac{ir}{e}\right)$ and $P_n^m\left(\frac{ir}{e}\right)$ in (38′) and (39′) contain r as a factor. Hence, since $\frac{1}{r}$ is a factor both of (38′) and (39′), these functions will be infinite when $r = 0$ and $\theta = \frac{1}{2}\pi$ unless B is zero. But A need not vanish. Now, for any point on the focal circle, $r = 0$ and $\theta = \frac{1}{2}\pi$; and, since this circle lies entirely within the ellipsoid $r = a$, for the solution to be valid throughout this region B must be zero. Thus the general solution for this region is

$$V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m\left(\frac{ir}{e}\right) T_n^m(\cos\theta)\{C_n^m \cos m\phi + D_n^m \sin m\phi\}. \tag{40}$$

§ 7. **Expression for the Reciprocal of the Distance between two Points.** The distance R between the two points whose elliptic co-ordinates of the type given in § 5 are (r, θ, ϕ), $(r,\ _1\theta_1,\ \phi_1)$ is given by the equation

$$\begin{aligned} R^2 = r^2 + r_1^2 + e^2(\sin^2\theta + \sin^2\theta_1) \\ - 2\sqrt{(r^2 + e^2)}\sqrt{(r_1^2 + e^2)}\sin\theta\sin\theta_1\cos(\phi - \phi_1) \\ - 2rr_1\cos\theta\cos\theta_1. \end{aligned}$$

If we assume that $r > r_1$, then, as in § 4, we find that

$$\frac{1}{R} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} H_n^m Q_n^m\left(\frac{ir}{e}\right) P_n^m\left(\frac{ir_1}{e}\right) T_n^m(\cos\theta) T_n^m(\cos\theta_1)\cos m(\phi - \phi_1).$$

If now e tends to zero, this equation leads to the identity

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{r_1^n}{r^{n+1}}\Bigg\{P_n(\cos\theta)P_n(\cos\theta_1) \\ + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi - \phi_1)\Bigg\} \\ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\operatorname*{Lim}_{e\to 0}\left\{H_n^m Q_n^m\left(\frac{ir}{e}\right)P_n^m\left(\frac{ir_1}{e}\right)\right\} \\ \times T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi - \phi_1), \end{aligned}$$

from which it follows that

$$\operatorname*{Lim}_{e\to 0}\left\{H_n^m Q_n^m\left(\frac{ir}{e}\right)P_n^m\left(\frac{ir_1}{e}\right)\right\} = 2\frac{(n-m)!}{(n+m)!}\frac{r_1^n}{r^{n+1}},$$

the factor 2 being omitted when $m = 0$. But,

$$\operatorname*{Lim}_{e\to 0}\left(\frac{e}{r_1}\right)^n P_n^m\left(\frac{ir_1}{e}\right) = \frac{(2n)!\, i^n}{2^n \,.\, n!\,(n-m)!},$$

and

$$\operatorname*{Lim}_{e\to 0}\left(\frac{r}{e}\right)^{n+1} Q_n^m\left(\frac{ir}{e}\right) = (-1)^m i^{-n-1}\frac{B(\frac{1}{2}, n+1)(n+m)!}{2^{n+1}\, n!},$$

so that

$$\operatorname*{Lim}_{e\to 0}\{H_n^m e\} = (-1)^m 2i(2n+1)\left\{\frac{(n-m)!}{(n+m)!}\right\}^2.$$

Hence, as in § 4, we find that

$$\frac{1}{R} = \frac{i}{e}\sum_{n=0}^{\infty}(2n+1)\Bigg[Q_n\left(\frac{ir}{e}\right)P_n\left(\frac{ir_1}{e}\right)P_n(\cos\theta)P_n(\cos\theta_1)$$

$$+ 2\sum_{m=1}^{n}(-1)^m\left\{\frac{(n-m)!}{(n+m)!}\right\}^2 Q_n^m\left(\frac{ir}{e}\right)P_n^m\left(\frac{ir_1}{e}\right)$$

$$\times\ T_n^m(\cos\theta)T_n^m(\cos\theta_1)\cos m(\phi - \phi_1)\Bigg]. \qquad (41)$$

The formulæ obtained in this and the previous section for the oblate ellipsoid of rotation can be employed in applications to potential problems just as were the corresponding formulæ for the prolate ellipsoid of revolution in §§ 3, 4.

The problems connected with ellipsoids all of whose axes are unequal can be treated in a similar manner, but the functions involved, known as Lamé's Functions, are much more complicated than the Legendre Functions.

CHAPTER XII

ECCENTRIC SPHERES

§ 1. **Dipolar Co-ordinates in Two Dimensions.** Consider the orthogonal systems of coaxal circles

$$x^2 + y^2 + c^2 = 2xc \coth t, \qquad . \quad . \quad . \quad (1)$$

and

$$x^2 + y^2 - c^2 = 2yc \cot u, \qquad . \quad . \quad . \quad (2)$$

where c is constant, and t and u are real variable parameters of the two systems. The first system is of the non-intersecting type, with the y-axis as radical axis; the equation (1) may be written

$$(x - c \coth t)^2 + y^2 = c^2 \operatorname{cosech}^2 t, \qquad . \quad . \quad (1')$$

so that the centre is the point $(c \coth t, 0)$ and the radius is $|c \operatorname{cosech} t|$: when t is positive the circle lies to the right of the origin, when t is negative, to the left. If $t = \pm \infty$, the radius is zero, and the corresponding centres, $L(c, 0)$ and $L'(-c, 0)$, are the limiting points of the system. As t varies from $+\infty$ to 0, the radius increases from 0 to $+\infty$, and the centre moves from L to $+\infty$ on the x-axis; while, as t varies from $-\infty$ to 0, the centre moves from L' to $-\infty$ on the x-axis, and the radius increases from 0 to ∞.

Example 1. If P is any point on circle (1), show that $PL/PL' = e^{-t}$.

Example 2. Show that L and L′ are inverse points with regard to circle (1).

Example 3. If A is the centre of circle (1), show that $AL/AL' = e^{-u}$.

The second coaxal system (2) is of the intersecting type, with the x-axis as radical axis: the common points of the system are $L(c, 0)$ and $L'(-c, 0)$, the limiting points of system (1). Equation (2) may be written

$$x^2 + (y - c \cot u)^2 = c^2 \operatorname{cosec}^2 u, \qquad . \quad . \quad (2')$$

so that the centre is $(c\cot u, 0)$ and the radius is $|c\operatorname{cosec} u|$. As

$$(c\cot u)^2 + (c\coth t)^2 = (c\operatorname{cosec} u)^2 + (c\operatorname{cosech} t)^2,$$

or *the square of the distance between the centres = the sum of the squares of the radii*, it follows that each circle of system (1) is orthogonal to each circle of system (2). When $u = \frac{1}{2}\pi$, the circle (2) has its centre at the origin, and its radius is c; when u is zero the radius is infinite, and the centre is at $+\infty$ on the y-axis, while if $u = \pi$ the radius is again infinite, and the centre is at $-\infty$ on the y-axis.

It is possible to express the co-ordinates of any point $P(x, y)$ in terms of t and u, the parameters of the circles of systems (1) and (2) which pass through that point. To do this, multiply (1) and (2) by $\tanh t$ and $\tan u$ respectively, square, and add; then

$$\{(x^2+y^2)^2+c^4\}(\tanh^2 t+\tan^2 u) + 2c^2(x^2+y^2)(\tanh^2 t - \tan^2 u) = 4c^2(x^2+y^2),$$

or

$$(x^2+y^2)^2 + c^4 - 2c^2(x^2+y^2)\frac{\cosh^2 t + \cos^2 u}{\cosh^2 t - \cos^2 u} = 0.$$

On solving this equation, it is found that

$$x^2 + y^2 = c^2\frac{(\cosh t + \cos u)^2}{\cosh^2 t - \cos^2 u} = c^2\frac{\cosh t + \cos u}{\cosh t - \cos u}, \tag{3}$$

or

$$x^2 + y^2 = c^2\frac{(\cosh t - \cos u)^2}{\cosh^2 t - \cos^2 u} = c^2\frac{\cosh t - \cos u}{\cosh t + \cos u}. \tag{4}$$

If now the value of $x^2 + y^2$ given in (3) be substituted in (1) and (2), it is found that

$$x = c\frac{\sinh t}{\cosh t - \cos u},\ y = c\frac{\sin u}{\cosh t - \cos u}, \tag{5}$$

while equation (4) leads to the values

$$x = c\frac{\sinh t}{\cosh t + \cos u},\ y = -c\frac{\sin u}{\cosh t + \cos u}. \tag{6}$$

These two sets of equations give the co-ordinates of the two points of intersection of the circles (1) and (2): the co-ordinates of the point (6) may be obtained from those of the point (5) by substituting $u - \pi$ for u: thus, by letting t vary from $-\infty$ to $+\infty$, and u from $-\pi$ to $+\pi$ in (5), the co-ordinates of every point in the plane are given once and once only. The variables (t, u) are called the *Dipolar Co-ordinates* of the point $P(x, y)$ given by the equations (5).

Example 4. If P is a point on that part of the circle (2) which lies above the x-axis, show that $\angle L'PL = u$, where $0 \leqq u \leqq \pi$, while if P lies on the lower part of the circle, $\angle L'PL = u - \pi$.

Example 5. Show that the equation

$$x + iy = c\frac{e^{t-iu} + 1}{e^{t-iu} - 1}$$

is equivalent to the pair of equations (5).

§ 2. **Dipolar Co-ordinates in Three Dimensions.** Now let the figure composed of the parts of the circles of systems (1) and (2) which lie above the x-axis be revolved about the x-axis: for this figure t varies from $-\infty$ to $+\infty$ and u from 0 to π. Then circles (1) trace out eccentric spheres whose equations are

$$x^2 + y^2 + z^2 + c^2 = 2xc \coth t;$$

the circles (2) trace out surfaces of revolution of degree 4 which intersect these spheres orthogonally; and a third system of surfaces, orthogonal to both these orthogonal systems, is given by the planes which pass through the x-axis. Then, if v be the angle turned through by the rotating plane, the co-ordinates of all points of space are given by the equations

$$x = c\frac{\sinh t}{\cosh t - \cos u}, \quad y = \frac{c \sin u \cos v}{\cosh t - \cos u}, \quad z = \frac{c \sin u \sin v}{\cosh t - \cos u}, \tag{7}$$

where t, u, v are the parameters of the surfaces through the point $P(x, y, z)$ and $-\infty \leqq t \leqq +\infty$, $0 \leqq u \leqq \pi$, $0 \leqq v \leqq 2\pi$. The variables t, u, v are *Three-dimensional Dipolar Co-ordinates.*

From equations (XI., 2) it follows that

$$l = m = \frac{c}{\cosh t - \cos u}, \quad n = \frac{c \sin u}{\cosh t - \cos u}. \tag{8}$$

Also, if R is the distance between the points whose rectangular and dipolar co-ordinates are (x, y, z), (x_1, y_1, z_1) and (t, u, v), (t_1, u_1, v_1) respectively,

$$\begin{aligned}
R^2 &= x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z_1^2 - 2(xx_1 + yy_1 + zz_1) \\
&= c^2\frac{\cosh t + \cos u}{\cosh t - \cos u} + c^2\frac{\cosh t_1 + \cos u_1}{\cosh t_1 - \cos u_1} \\
&\qquad - 2c^2\frac{\sinh t \sinh t_1 + \sin u \sin u_1 \cos(v - v_1)}{(\cosh t - \cos u)(\cosh t_1 - \cos u_1)} \\
&= 2c^2\frac{\cosh t \cosh t_1 - \sinh t \sinh t_1 - \cos\gamma}{(\cosh t - \cos u)(\cosh t_1 - \cos u_1)},
\end{aligned}$$

where $\cos\gamma = \cos u \cos u_1 + \sin u \sin u_1 \cos(v - v_1)$.

But

$$\begin{aligned}
\cosh t \cosh t_1 - \sinh t \sinh t_1 &= \cosh(t - t_1) \\
&= \tfrac{1}{2}\{e^{t - t_1} + e^{-(t - t_1)}\},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\cosh t \cosh t_1 - \sinh t \sinh t_1 - \cos\gamma \\
&\qquad = \tfrac{1}{2}e^{t - t_1}\{1 - 2e^{-(t - t_1)}\cos\gamma + e^{-2(t - t_1)}\}
\end{aligned}$$

or

$$= \tfrac{1}{2}e^{-(t - t_1)}\{1 - 2e^{t - t_1}\cos\gamma + e^{2(t - t_1)}\}.$$

Thus

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{c}\,\frac{\sqrt{(\cosh t - \cos u)}\sqrt{(\cosh t_1 - \cos u_1)}}{e^{\pm\frac{1}{2}(t - t_1)}} \\
&\qquad \times \frac{1}{\sqrt{\{1 - 2e^{\mp(t - t_1)}\cos\gamma + e^{\mp 2(t - t_1)}\}}},
\end{aligned}$$

the upper sign being taken if $t > t_1$, the lower if $t < t_1$; in the first case $e^{-(t - t_1)} < 1$, in the second $e^{t - t_1} < 1$, so that

$$\frac{1}{\sqrt{\{1 - 2e^{\mp(t - t_1)}\cos\gamma + e^{\mp 2(t - t_1)}\}}} = \sum_{n=0}^{\infty} e^{\mp n(t - t_1)}P_n(\cos\gamma),$$

and

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{c}\sqrt{(\cosh t - \cos u)}\sqrt{(\cosh t_1 - \cos u_1)} \\
&\qquad \times \sum_{n=0}^{\infty} e^{\mp(n + \frac{1}{2})(t - t_1)}P_n(\cos\gamma). \qquad (9)
\end{aligned}$$

§ 3. **The Problem of Two Eccentric Spheres.** The surface-distribution on two electrically charged insulated conducting spheres external to each other, and not subjected to any external forces, will now be considered. Let the line of centres be taken as x-axis, with the origin O at its point of intersection with the radical plane of the spheres; A and B denote the centres of the spheres to the right and left of O respectively, and L and L′ are the limiting points of the system. Then the dipolar co-ordinates are given by (7), where c = OL, while the values t_1 and t_2 of t which correspond to the spheres whose centres are A and B are given (§ 1, *ex.* 3) by $\mathrm{AL}/\mathrm{AL}' = e^{-2t_1}$ and $\mathrm{BL}/\mathrm{BL}' = e^{-2t_2}$, t_1 of course being positive, t_2 negative.

Let (u_1, v_1) be the dipolar co-ordinates of a point on the sphere t_1, (u_2, v_2) of a point on t_2, $\sigma(u_1, v_1)$, $\sigma(u_2, v_2)$ the densities at these points, V and W the potentials at (t, u, v) of the surface distributions on t_1 and t_2 respectively. Then

$$\mathrm{V} = \iint \frac{\sigma(u, v)}{\mathrm{R}} d\mathrm{S},$$

the integral being taken over the surface of t_1; or, by (XI., 3″) and (8)

$$\mathrm{V} = c^2 \int_0^{2\pi} \int_0^{\pi} \frac{\sigma(u_1, v_1) \sin u_1 \, du_1 dv_1}{(\cosh t_1 - \cos u_1)^2} \frac{1}{\mathrm{R}}.$$

Now assume that

$$\frac{\sigma(u_1, v_1)}{(\cosh t_1 - \cos u_1)^{\frac{3}{2}}} = \sum_{m=0}^{\infty} \mathrm{X}_m(u_1, v_1), \quad . \quad . \quad (10)$$

where $\mathrm{X}_m(u_1, v_1)$ is a surface harmonic of degree m; and for 1/R employ the formula (9) in which, if the point (t, u, v) lies inside the sphere A, $t > t_1$, while if it lies outside, $t < t_1$. Then, for an internal point

$$\mathrm{V}_i = c\sqrt{(\cosh t - \cos u)} \int_0^{2\pi} \int_0^{\pi} \sin u_1 \, du_1 dv_1 \left\{ \sum_{m=0}^{\infty} \mathrm{X}_m(u_1, v_1) \right\}$$

$$\times \left\{ \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})(t-t_1)} \mathrm{P}_n(\cos \gamma) \right\}.$$

If these series are multiplied, and the product integrated term by term, it follows from (VII., 29) and (VII., 33) that

$$\mathrm{V}_i = 4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(n+\frac{1}{2})(t-t_1)} \mathrm{X}_n(u, v); \quad (11)$$

and similarly, for (t, u, v) an external point,

$$V_e = 4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(n+\frac{1}{2})(t_1 - t)} X_n(u, v). \tag{12}$$

Again, for an internal point of the sphere B, since t_2 is negative, $t < t_2$, while, for an external point, $t > t_2$: hence, if

$$\frac{\sigma(u_2, v_2)}{(\cosh t_2 - \cos u_2)^{\frac{3}{2}}} = \sum_{m=0}^{\infty} Y_m(u_2, v_2), \tag{13}$$

$$W_i = 4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{(n+\frac{1}{2})(t - t_2)} Y_n(u, v), \tag{14}$$

and

$$W_e = 4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{(n+\frac{1}{2})(t_2 - t)} Y_n(u, v). \tag{15}$$

Since the potentials of the spheres A and B are constant, say h and k, it follows that

$V_i + W_e = h$, $(t > t_1)$, and $V_e + W_i = k$, $(t < t_2)$; or, if $t > t_1$,

$$4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} \begin{Bmatrix} e^{-(n+\frac{1}{2})(t - t_1)} X_n(u, v) \\ + e^{(n+\frac{1}{2})(t_2 - t)} Y_n(u, v) \end{Bmatrix} = h, \tag{16}$$

and, if $t < t_2$,

$$4\pi c\sqrt{(\cosh t - \cos u)} \sum_{n=0}^{\infty} \frac{1}{2n+1} \begin{Bmatrix} e^{-(n+\frac{1}{2})(t_1 - t)} X_n(u, v) \\ + e^{(n+\frac{1}{2})(t - t_2)} Y_n(u, v) \end{Bmatrix} = k. \tag{17}$$

In order to determine the functions X_n, Y_n by means of these equations, we divide by $\sqrt{(\cosh t - \cos u)}$ and express the reciprocal of this function in terms of Legendre Coefficients: thus

$$\frac{1}{\sqrt{(\cosh t - \cos u)}} = \frac{\sqrt{2}}{\sqrt{(e^t + e^{-t} - 2\cos u)}}$$

$$= \frac{\sqrt{2} \,.\, e^{-\frac{1}{2}t}}{\sqrt{(1 - 2e^{-t}\cos u + e^{-2t})}}, \tag{18}$$

or

$$= \frac{\sqrt{2} \,.\, e^{\frac{1}{2}t}}{\sqrt{(1 - 2e^{t}\cos u + e^{2t})}}, \tag{19}$$

the expression (18) being employed in (16), since there $t > 0$, so that $e^{-t} < 1$, and (19) being applied to (17), since when $t < 0$, $e^{t} < 1$. From (16) and (18) it follows that

$$4\pi c \sum_{n=0}^{\infty} \frac{1}{2n+1}\{e^{-(n+\frac{1}{2})(t-t_1)}X_n(u, v) + e^{(n+\frac{1}{2})(t_2-t)}Y_n(u, v)\}$$

$$= h\sqrt{2} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})t} P_n(\cos u), \quad . \quad . \quad (20)$$

and from (17) and (19) that

$$4\pi c \sum_{n=0}^{\infty} \frac{1}{2n+1}\{e^{-(n+\frac{1}{2})(t_1-t)}X_n(u, v) + e^{(n+\frac{1}{2})(t-t_2)}Y_n(u, v)\}$$

$$= k\sqrt{2} \sum_{n=0}^{\infty} e^{(n+\frac{1}{2})t}P_n(\cos u). \quad . \quad . \quad (21)$$

If now the corresponding terms in the identities (20) and (21) be equated, the resulting equations are

$$\frac{4\pi c}{2n+1}\{e^{(n+\frac{1}{2})t_1}X_n(u, v) + e^{(n+\frac{1}{2})t_2}Y_n(u, v)\} = h\sqrt{2}P_n(\cos u), \quad (22)$$

$$\frac{4\pi c}{2n+1}\{e^{-(n+\frac{1}{2})t_1}X_n(u, v) + e^{-(n+\frac{1}{2})t_2}Y_n(u, v)\} = k\sqrt{2}P_n(\cos u), \quad (23)$$

which, on solution, give

$$\frac{4\pi c}{2n+1} X_n(u, v) = \frac{\sqrt{2}\, P_n(\cos u)\{h\, e^{-(n+\frac{1}{2})t_2} - k\, e^{(n+\frac{1}{2})t_2}\}}{2 \sinh\{(n+\frac{1}{2})(t_1 - t_2)\}}, \quad (24)$$

and

$$\frac{4\pi c}{2n+1} Y_n(u, v) = \frac{\sqrt{2}\, P_n(\cos u)\{k\, e^{(n+\frac{1}{2})t_1} - h\, e^{-(n+\frac{1}{2})t_1}\}}{2 \sinh\{(n+\frac{1}{2})(t_1 - t_2)\}}. \quad (25)$$

By substituting these values of X_n and Y_n in (10) and (13), $\sigma(u_1, v_1)$ and $\sigma(u_2, v_2)$ are obtained in terms of h and k: the constants h and k can then be determined by integration if the total charges on the spheres are known.

Note 1. If the sphere B is earthed, $k = 0$.

Note 2. If $t_2 = 0$, the sphere B becomes the plane $x = 0$, so that the corresponding formulæ give the solutions for the case of a sphere and a plane.

§ 4. **Solution of Laplace's Equation.** From (8) and (XI., 5) it follows that, for dipolar co-ordinates,

$$\nabla^2 V = \frac{(\cosh t - \cos u)^3}{c^3 \sin u} \times \left[\frac{\partial}{\partial t}\left\{\frac{c \sin u}{\cosh t - \cos u}\frac{\partial V}{\partial t}\right\} + \frac{\partial}{\partial u}\left\{\frac{c \sin u}{\cosh t - \cos u}\frac{\partial V}{\partial u}\right\} + \frac{c}{\sin u(\cosh t - \cos u)}\frac{\partial^2 V}{\partial v^2} \right]. \quad (26)$$

Here let $V = V_1\sqrt{(\cosh t - \cos u)}$,

then

$$\frac{\partial}{\partial t}\left\{\frac{1}{\cosh t - \cos u}\frac{\partial V}{\partial t}\right\}$$
$$= \frac{\partial}{\partial t}\left\{\frac{1}{\sqrt{(\cosh t - \cos u)}}\frac{\partial V_1}{\partial t} + \frac{\sinh t}{2(\cosh t - \cos u)^{\frac{3}{2}}}V_1\right\}$$
$$= \frac{1}{\sqrt{(\cosh t - \cos u)}} \times \left[\frac{\partial^2 V_1}{\partial t^2} + \left\{\frac{\cosh t}{2(\cosh t - \cos u)} - \frac{3 \sinh^2 t}{4(\cosh t - \cos u)^2}\right\}V_1\right],$$

and

$$\frac{1}{\sin u}\frac{\partial}{\partial u}\left\{\frac{\sin u}{\cosh t - \cos u}\frac{\partial V}{\partial u}\right\}$$
$$= \frac{1}{\sin u}\frac{\partial}{\partial u}\left\{\frac{\sin u}{\sqrt{(\cosh t - \cos u)}}\frac{\partial V_1}{\partial u} + \frac{\sin^2 u}{2(\cosh t - \cos u)^{\frac{3}{2}}}V_1\right\}$$
$$= \frac{1}{\sqrt{(\cosh t - \cos u)}}\left[\frac{1}{\sin u}\frac{\partial}{\partial u}\left\{\sin u\frac{\partial V_1}{\partial u}\right\} + \left\{\frac{\cos u}{\cosh t - \cos u} - \frac{3 \sin^2 u}{4(\cosh t - \cos u)^2}\right\}V_1\right],$$

that

$$\nabla^2 V = \frac{(\cosh t - \cos u)^{\frac{5}{2}}}{c^2}\left[\frac{\partial^2 V_1}{\partial t^2} + \frac{1}{\sin u}\frac{\partial}{\partial u}\left\{\sin u\frac{\partial V_1}{\partial u}\right\} + \frac{1}{\sin^2 u}\frac{\partial^2 V_1}{\partial v^2} - \frac{1}{4}V_1\right], \quad (27)$$

and Laplace's Equation becomes

$$\frac{\partial^2 V_1}{\partial t^2} + \frac{1}{\sin u}\frac{\partial}{\partial u}\left\{\sin u\frac{\partial V_1}{\partial u}\right\} + \frac{1}{\sin^2 u}\frac{\partial^2 V_1}{\partial v^2} - \frac{1}{4}V_1 = 0. \quad (28)$$

Now let $V_1 = W_1W_2W_3$, where W_1, W_2, W_3 are respectively functions of t, u, v alone; then, from (28),

$$\frac{1}{W_1}\frac{d^2W_1}{dt^2} + \frac{1}{W_2}\frac{1}{\sin u}\frac{d}{du}\left\{\sin u\frac{dW_2}{du}\right\} + \frac{1}{\sin^2 u}\frac{1}{W_3}\frac{d^2W_3}{dv^2} - \frac{1}{4} = 0. \tag{29}$$

In this equation the first two terms and the last term are independent of v, and therefore so is also the third term; thus the value of $\frac{1}{W_3}\frac{d^2W_3}{dv^2}$ must be constant, so that

$$\frac{d^2W_3}{dv^2} = CW_3;$$

and therefore

$$W_3 = A\cos(v\sqrt{-C}) + B\sin(v\sqrt{-C}).$$

But W_3 is periodic, of period 2π; hence $\sqrt{(-C)}$ must be an integer, and we may write

$$W_3 = A\cos(mv) + B\sin(mv),$$

where m is a positive integer.

Again, in (29), replacing $\frac{1}{W_3}\frac{d^2W_3}{dv^2}$ by its value $-m^2$, and noting that the first term is independent of u, we see that

$$\frac{1}{W_2}\frac{1}{\sin u}\frac{d}{du}\left\{\sin u\frac{dW_2}{du}\right\} - \frac{m^2}{\sin^2 u}$$

is constant. Denoting this constant by $-\alpha(\alpha+1)$, we obtain the equation

$$\frac{1}{\sin u}\frac{d}{du}\left\{\sin u\frac{dW_2}{du}\right\} + \left\{\alpha(\alpha+1) - \frac{m^2}{\sin^2 u}\right\}W_2 = 0,$$

which, when $\cos u$ is taken as independent variable, becomes Legendre's Associated Equation. The general solution

$$W_2 = CT_\alpha^m(\cos u) + DQ_\alpha^m(\cos u)$$

must be finite for $u = 0$ and $u = \pi$, and therefore D must be zero and α must be an integer. Hence, if $\alpha = n$,

$$W_2W_3 = T_n^m(\cos u)(A\cos mv + B\sin mv).$$

Equation (29) now reduces to

$$\frac{d^2W_1}{dt^2} - \{n(n+1) + \tfrac{1}{4}\}W_1 = 0,$$

of which the general solution is

$$W_1 = Ee^{(n+\frac{1}{2})t} + Fe^{-(n+\frac{1}{2})t}.$$

The most general solution of (29) obtained in this way is

$$V_1 = \sum_{n=0}^{\infty} \{e^{(n+\frac{1}{2})t}X_n(u, v) + e^{-(n+\frac{1}{2})t}Y_n(u, v)\}, \qquad (30)$$

where X_n and Y_n are arbitrary surface harmonics of degree n.

For a space which contains the point L, where $t = +\infty$, $X_n(u, v)$ must be zero, while, for a space which includes L′, where $t = -\infty$, $Y_n(u, v)$ must vanish.

Boundary Problems.—In order to obtain the potential function V for the space bounded by the spheres $t = t_1$ and $t = t_2$ when the values of V on these spheres are given to be $f(u, v)$ and $\phi(u, v)$ respectively, expand $f(u, v)/\sqrt{(\cosh t - \cos u)}$ and $\phi(u, v)/\sqrt{(\cosh t - \cos u)}$ in terms of surface harmonics, and, in the first case equate the expression so obtained to (30), with $t = t_1$, in the second case to (30) with $t = t_2$. From the two resulting identities $X_n(u, v)$ and $Y_n(u, v)$ may be determined. This method of solution holds either when one sphere encloses the other or when the spheres are external to each other. In the latter case the region extends to infinity: but the condition that the potential should vanish at infinity is fulfilled in consequence of the fact that, at infinity, $t = 0$ and $u = 0$, so that $\sqrt{(\cosh t - \cos u)} = 0$.

To solve this problem for the interior of a sphere, say sphere A, within which t is positive, put every $X_n(u, v) = 0$.

Other problems connected with pairs of spheres can be solved in a similar manner.

§ 5. **Ring Surfaces.** If the figure discussed in § 1, instead of being revolved about the x-axis, be revolved about the y-axis, the circles $u = constant$ generate spheres, the circles $t = constant$ generate anchor rings. These surfaces, with the planes through the y-axis, form three orthogonal systems of surfaces, and the co-ordinates of any point in space are given by the equations, derived from (5),

$$x = c\frac{\sinh t \cos v}{\cosh t - \cos u}, \quad y = c\frac{\sin u}{\cosh t - \cos u}, \quad z = c\frac{\sinh t \sin v}{\cosh t - \cos u}, \qquad (31)$$

where v is the angle turned through by the rotating plane. Since it is only necessary to revolve the part of the (x, y) plane for which x is positive, t varies only from 0 to $+\infty$, u varies from $-\pi$ to $+\pi$, and v varies from 0 to 2π. From (XI., 2) it follows that

$$l = m = \frac{c}{\cosh t - \cos u}, \quad n = \frac{c \sinh t}{\cosh t - \cos u} \text{ .} \tag{32}$$

Thus, from (XI., 5),

$$\nabla^2 V = \frac{(\cosh t - \cos u)^3}{c^2 \sinh t}$$
$$\times \left[\frac{\partial}{\partial t}\left\{\frac{\sinh t}{\cosh t - \cos u}\frac{\partial V}{\partial t}\right\} + \frac{\partial}{\partial u}\left\{\frac{\sinh t}{\cosh t - \cos u}\frac{\partial V}{\partial u}\right\} + \frac{1}{\sinh t\,(\cosh t - \cos u)}\frac{\partial^2 V}{\partial v^2} \right].$$

Here put $V = V_1\sqrt{(\cosh t - \cos u)}$, and then

$$\frac{\partial}{\partial t}\left\{\frac{\sinh t}{\cosh t - \cos u}\frac{\partial V}{\partial t}\right\} = \frac{\partial}{\partial t}\left\{\frac{\sinh t}{\sqrt{(\cosh t - \cos u)}}\frac{\partial V_1}{\partial t} + \frac{\sinh^2 t}{2(\cosh t - \cos u)^{\frac{3}{2}}}V_1\right\}$$
$$= \frac{1}{\sqrt{(\cosh t - \cos u)}}\left\{\sinh t\frac{\partial^2 V_1}{\partial t^2} + \cosh t\frac{\partial V_1}{\partial t} + \frac{\sinh t \cosh t}{\cosh t - \cos u}V_1 - \frac{3 \sinh^3 t}{4(\cosh t - \cos u)^2}V_1\right\},$$

$$\frac{\partial}{\partial u}\left\{\frac{\sinh t}{\cosh t - \cos u}\frac{\partial V}{\partial u}\right\} = \frac{\partial}{\partial u}\left\{\frac{\sinh t}{\sqrt{(\cosh t - \cos u)}}\frac{\partial V_1}{\partial u} + \frac{\sinh t \sin u}{2(\cosh t - \cos u)^{\frac{3}{2}}}V_1\right\}$$
$$= \frac{1}{\sqrt{(\cosh t - \cos u)}}\left\{\sinh t\frac{\partial^2 V_1}{\partial u^2} + \frac{\sinh t \cos u}{2(\cosh t - \cos u)}V_1 - \frac{3}{4}\frac{\sinh t \sin^2 u}{(\cosh t - \cos u)^2}V_1\right\},$$

so that

$$\nabla^2 V = \frac{(\cosh t - \cos u)^{\frac{5}{2}}}{c^2}$$
$$\times \left[\frac{1}{\sinh t}\frac{\partial}{\partial t}\left\{\sinh t\frac{\partial V_1}{\partial t}\right\} + \frac{\partial^2 V_1}{\partial u^2} + \frac{1}{\sinh^2 t}\frac{\partial^2 V_1}{\partial v^2} + \frac{1}{4}V_1\right].$$

Thus Laplace's Equation becomes

$$\frac{1}{\sinh t}\frac{\partial}{\partial t}\left\{\sinh t\,\frac{\partial V_1}{\partial t}\right\}+\frac{\partial^2 V_1}{\partial u^2}+\frac{1}{\sinh^2 t}\frac{\partial^2 V_1}{\partial v^2}+\frac{1}{4}V_1=0. \quad (33)$$

Here put $V_1 = W_1W_2W_3$, where W_1, W_2, W_3 are functions respectively of t, u, v alone; then, as in § 4, we find that

$$W_2 = A\cos nu + B\sin nu,$$
$$W_3 = C\cos mv + D\sin mv,$$

where m and n are integers. Equation (33) then reduces to

$$\frac{1}{\sinh t}\frac{1}{W_1}\frac{d}{dt}\left\{\sinh t\frac{dW_1}{dt}\right\}-n^2-\frac{m^2}{\sinh^2 t}+\frac{1}{4}=0,$$

or, if $\lambda = \cosh t$,

$$\frac{d}{d\lambda}\left\{(\lambda^2-1)\frac{dW_1}{d\lambda}\right\}-\left\{(n-\tfrac{1}{2})(n+\tfrac{1}{2})+\frac{m^2}{\lambda^2-1}\right\}=0,$$

which is Legendre's Associated Equation with $n - \frac{1}{2}$ in place of n. Thus its general solution is

$$W_1 = EP^m_{n-\frac{1}{2}}(\lambda) + FQ^m_{n-\frac{1}{2}}(\lambda).$$

The functions $P^m_{n-\frac{1}{2}}(\lambda)$ and $Q^m_{n-\frac{1}{2}}(\lambda)$ are sometimes described as *Ring Functions*

CHAPTER XIII

CLERK MAXWELL'S THEORY OF SPHERICAL HARMONICS

§ 1. **Theorem.** If $f_n(x, y, z)$ is a homogeneous rational integral function of degree n, the function

$$\left[1-\frac{r^2}{2(2n-1)}\nabla^2+\frac{r^4}{2.4(2n-1)(2n-3)}\nabla^4-\dots\right]f_n(x,y,z), \quad (1)$$

where $\nabla^{2r}\equiv\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^r$, is a solid harmonic of degree n.

Let $w = \Phi(x, y, z)$ be any function of x, y, z, and let $F(w)$ be any function of w: then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)F(w)=\chi_0\frac{d^nF}{dw^n}+\chi_1\frac{d^{n-1}F}{dw^{n-1}}+\dots+\chi_{n-1}\frac{dF}{dw}, \quad (2)$$

where $\chi_0, \chi_1, \dots, \chi_{n-1}$ are functions of x, y, z which are independent of the nature of the function $F(w)$.

Now for $F(w)$ take the function w^s, where s is an integer $\leqq n$; then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)w^s = s!\,\chi_{n-s}+\frac{s!}{1!}\chi_{n-s+1}w+\dots$$
$$+\frac{s!}{(s-1)!}\chi_{n-1}w^{s-1}. \quad (3)$$

But, if $w = \Phi(x, y, z)$,

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\{\Phi(x,y,z)\}^s=f_n\left(\frac{\partial}{\partial h},\frac{\partial}{\partial k},\frac{\partial}{\partial l}\right)\{\Phi(x+h,y+k,z+l)\}^s,$$

where h, k, l are all put equal to zero after the operation has been carried out: also

$$\{\phi(x+h,y+k,z+l)\}^s =$$
$$[\Phi(x,y,z)+\{\Phi(x+h,y+k,z+l)-\Phi(x,y,z)\}]^s$$
$$=\sum_{t=0}^{s}\frac{s!}{t!\,(s-t)!}\{\Phi(x,y,z)\}^t\{\Phi(x+h,y+k,z+l)-\Phi(x,y,z)\}^{s-t}.$$

Hence

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)w^s = \frac{s!}{s!}u_{n-s} + \frac{s!}{1!(s-1)!}wu_{n-s+1} + \ldots$$
$$+ \frac{s!}{(s-1)!\,1!}w^{s-1}u_{n-1}, \quad (4)$$

where

$$u_{n-t} = f_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\{\Phi(x+h, y+k, z+l) - \Phi(x, y, z)\}^t,$$

and h, k, l finally become zero.

Now compare equations (3) and (4) for $s = 1, 2, 3, \ldots, n$; we find that

$$1!\,\chi_{n-1} = \frac{1!}{1!}u_{n-1},$$

$$2!\,\chi_{n-2} + \frac{2!}{1!}\chi_{n-1}w = \frac{2!}{2!}u_{n-2} + \frac{2!}{1!\,1!}u_{n-1}w,$$

$$3!\,\chi_{n-3} + \frac{3!}{1!}\chi_{n-2}w + \frac{3!}{2!}\chi_{n-1}w^2$$
$$= \frac{3!}{3!}u_{n-3} + \frac{3!}{1!\,2!}u_{n-2}w + \frac{3!}{2!\,1!}u_{n-1}w^2,$$

. . . .

Hence, from the first equation, $\chi_{n-1} = \frac{1}{1!}u_{n-1}$, from the second, $\chi_{n-2} = \frac{1}{2!}u_{n-2}$, from the third, $\chi_{n-3} = \frac{1}{3!}u_{n-3}$, and so on: thus

$$\chi_t = \frac{1}{(n-t)!}u_t$$
$$= \frac{1}{(n-t)!}f_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\{\Phi(x+h, y+k, z+l) - \Phi(x,y,z)\}^{n-t},$$

where h, k, l finally become zero.

Again, let $\Phi(x, y, z) = x^2 + y^2 + z^2$; then

$$\chi_t = \frac{1}{(n-t)!}f_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\{2(hx+ky+lz) + (h^2+k^2+l^2)\}^{n-t},$$
$$(h, k, l = 0),$$
$$= \frac{1}{(n-t)!}f_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\frac{(n-t)!}{(n-2t)!\,t!}$$
$$\times \{2(hx+ky+lz)\}^{n-2t}(h^2+k^2+l^2)^t, \quad (h, k, l, = 0)$$

since the only term which gives a non-zero value is the one that is a function of h, k, l of degree n.

But, if $\psi_n(h, k, l)$ is a homogeneous rational integral function of h, k, l, of degree n,

$$f_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\psi_n(h, k, l) = \psi_n\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)f_n(h, k, l);$$

hence

$$\chi_t = \frac{2^{n-2t}}{(n-2t)!\, t!}\left(x\frac{\partial}{\partial h} + y\frac{\partial}{\partial k} + z\frac{\partial}{\partial l}\right)^{n-2t}\left(\frac{\partial^2}{\partial h^2} + \frac{\partial^2}{\partial k^2} + \frac{\partial^2}{\partial l^2}\right)^t f_n(h, k, l).$$

Also,

$$\left(x\frac{\partial}{\partial h} + y\frac{\partial}{\partial k} + z\frac{\partial}{\partial l}\right)^s \psi_s(h, k, l) = s!\ \psi_s(x, y, z);$$

for, if one term of $\psi_s(h, k, l)$ is $Ch^p k^q l^{s-p-q}$, the only term of $\left(x\frac{\partial}{\partial h} + y\frac{\partial}{\partial k} + z\frac{\partial}{\partial l}\right)^s$, which, when applied to it, gives a non-zero value is

$$\frac{s!}{p!\, q!\, (s-p-q)!}x^p y^q z^{s-p-q}\frac{\partial^p}{\partial h^p}\frac{\partial^q}{\partial k^q}\frac{\partial^{s-p-q}}{\partial l^{s-p-q}}.$$

Therefore, since

$$\left(\frac{\partial^2}{\partial h^2} + \frac{\partial^2}{\partial k^2} + \frac{\partial^2}{\partial l^2}\right)^t f_n(h, k, l)$$

is a homogeneous rational integral function of degree $n - 2t$,

$$\chi_t = \frac{2^{n-2t}}{t!}\nabla^{2t} f_n(x, y, z),$$

where $$\nabla^{2t} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^t.$$

Next put this value of χ_t in (2), where $w = x^2 + y^2 + z^2 = r^2$, and in place of $F(r^2)$ write $\phi(r)$; thus

$$\begin{aligned} f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi(r) &= 2^n\frac{d^n\phi(r)}{d(r^2)^n}f_n(x, y, z) \\ &+ \frac{2^{n-2}}{1!}\frac{d^{n-1}\phi(r)}{d(r^2)^{n-1}}\nabla^2 f_n(x, y, z) \\ &+ \frac{2^{n-4}}{2!}\frac{d^{n-2}\phi(r)}{d(r^2)^{n-2}}\nabla^4 f_n(x, y, z) + \cdots \qquad (5) \end{aligned}$$

Finally, replacing $\phi(r)$ by r^{-1}, we find that

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = (-1)^n\frac{(2n)!}{2^n . n!}\frac{1}{r^{2n+1}}$$
$$\times\left[1 - \frac{r^2}{2(2n-1)}\nabla^2 + \frac{r^4}{2 . 4(2n-1)(2n-3)}\nabla^4 - \ldots\right]f_n(x,y,z). \tag{6}$$

But the function on the left-hand side of this equation is a solid harmonic of degree $-n-1$. Therefore, multiplying the equation by r^{2n+1}, we see that the expression (1) is a solid harmonic of degree n.

Corollary. From (5) it follows that, if $Y_n(x, y, z)$ is a solid harmonic of positive integral degree n,

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi(r) = 2^n\frac{d^n\phi(r)}{d(r^2)^n}Y_n(x, y, z),$$

and, in particular, that

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = (-1)^n\frac{(2n)!}{2^n . n!}\frac{1}{r^{2n+1}}Y_n(x, y, z). \tag{7}$$

§ 2. **Poles of a Spherical Harmonic.** Consider a sphere of radius r with the origin O as centre and let R be a point on the sphere; the direction OR may be called an *Axis*, and the point R is then known as the *Pole* of the axis.

If the direction-cosines of OR are (l_i, m_i, n_i), the axis OR is denoted by h_i. If, moreover, P is any point (x, y, z) on the sphere, the cosine of the angle POR is denoted by λ_i, so that

$$\lambda_i = \frac{l_i x + m_i y + n_i z}{r} = \cos \text{PR}, \quad . \quad . \tag{8}$$

where PR is written for the angle POR. The cosine of the angle between two axes h_i and h_j is denoted by μ_{ij}, so that

$$\mu_{ij} = l_i l_j + m_i m_j + n_i n_j \quad . \quad . \quad . \tag{9}$$

Finally, the operation $l_i\frac{\partial}{\partial x} + m_i\frac{\partial}{\partial y} + n_i\frac{\partial}{\partial z}$ is called "differentiation with respect to the axis h_i" and is denoted by $\frac{\partial}{\partial h_i}$. The symbol

$$\frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n}$$

represents differentiation by n of these operators in succession.

Potentials due to Singular Points. The potential at the point $P(x, y, z)$ due to a point charge e_0 of electricity at the origin is

$$V_0 = e_0 \frac{1}{r};$$

this function satisfies Laplace's equation, and, consequently, so does every function formed from it by differentiating with regard to any number of axes in succession. A point at which a point-charge e_0 is placed is called a *Singular Point of degree zero and strength* e_0.

To form a singular point of degree one at the origin a charge $-e_0$ is placed there, and a second charge $+e_0$ is placed at that point on the axis h_1 whose distance from the origin is α_0. The product $e_0\alpha_0$ is denoted by e_1, and this quantity is kept constant, while α_0 tends to zero and e_0 tends to infinity. The origin is then a *Singular Point of the first degree of strength* e_1 *and axis* h_1.

Similarly by taking singular points of degree one and strengths $-e_1$ and e_1 at the origin and at the point on the axis h_2 distant α_1 from the origin respectively and making α_1 tend to zero and e_1 to infinity so that $e_1\alpha_1 = e_2$, where e_2 is constant, a *Singular Point of the second degree of strength* e_2 *and axes* h_1 *and* h_2 is obtained. This process can obviously be continued indefinitely.

Now assume that

$$V_{s-1} = e_{s-1}\phi_{s-1}(x, y, z)$$

is the potential at P due to a singular point at the origin of strength e_{s-1}, degree $s - 1$, and axes $h_1, h_2, \ldots, h_{s-1}$; then a singular point of degree s and strength $e_s = e_{s-1}\alpha_{s-1}$ can be formed by taking a singular point of strength $-e_{s-1}$ at the origin O, and a singular point of strength e_{s-1} at a point Q distant α_{s-1} from O along the new axis h_s, and keeping e_s or $\alpha_{s-1}e_{s-1}$ constant while α_{s-1} tends to zero. The potential at P due to the singular point of degree $s - 1$ at O is

$$-e_{s-1}\phi_{s-1}(x, y, z),$$

and that due to the singular point at Q is

$$\begin{aligned} &e_{s-1}\phi_{s-1}(x_P - x_Q, y_P - y_Q, z_P - z_Q) \\ &\qquad = e_{s-1}\phi_{s-1}(x - l_s\alpha_{s-1}, y - m_s\alpha_{s-1}, z - n_s\alpha_{s-1}). \end{aligned}$$

Hence the potential at P due to the singular point of degree s at O is

$$\lim_{\alpha_{s-1}\to 0}\left\{\begin{array}{c} e_{s-1}\phi_{s-1}(x - l_s\alpha_{s-1},\ y - m_s\alpha_{s-1},\ z - n_s\alpha_{s-1}) \\ - e_{s-1}\phi_{s-1}(x, y, z). \end{array}\right\}$$

$$= - e_s\left(l_s\frac{\partial}{\partial x} + m_s\frac{\partial}{\partial y} + n_s\frac{\partial}{\partial z}\right)\phi_{s-1}(x, y, z)$$

$$= - e_s\frac{\partial}{\partial h_s}\phi_{s-1}(x, y, z).$$

But $\phi_0 = \frac{1}{r}$; therefore

$$V_s = (-1)^s e_s \frac{\partial^s}{\partial h_1 \partial h_2 \ldots \partial h_s}\frac{1}{r}$$

is the potential at the point (x, y, z) due to the singular point at the origin of strength e_s, order s, and axes $h_1, h_2, \ldots, h_r$

The function V_s is a solid harmonic of degree $- s - 1$; it is found convenient to express it in the form

$$V_s = s!\, e_s\frac{Y_s}{r^{s+1}},$$

where Y_s is a surface harmonic of degree s; thus

$$Y_s = \frac{(-1)^s}{s!}r^{s+1}\frac{\partial^s}{\partial h_1 \partial h_2 \ldots \partial h_s}\frac{1}{r}. \qquad (10)$$

Y_s is an expression involving the angles which the axes $h_1, h_2, \ldots, h_s$ make with the co-ordinate axes.

The *Poles* of the spherical harmonic Y_s are the poles of its axes.

Expressions for the Surface Harmonics. From (6) and (10)

$$Y_s = \frac{(2s)!}{2^s \cdot (s!)^2}\frac{1}{r^s}\left[1 - \frac{r^2}{2(2s-1)}\nabla^2 + \frac{r^4}{2 \cdot 4(2s-1)(2s-3)}\nabla^4 - \ldots\right]$$
$$\times \prod_{p=1}^{s}(l_px + m_py + n_pz), \qquad (11)$$

where
$$Y_s = \frac{(-1)^s}{s!}r^{s+1}\frac{\partial^s}{\partial h_1 \partial h_2 \ldots \partial h_s}\frac{1}{r}.$$

Now let $\Sigma(\mu^p\lambda^{s-2p})$ denote the sum of all possible products of p μ's and $(s - 2p)$ λ's [cf. (8) and (9)] taken from the μ's and λ's arising from the axes $h_1, h_2, \ldots, h_s$; in each product every suffix occurs once and once only. Then

$$\prod_{p=1}^{s}(l_px + m_py + n_pz) = r^s\Sigma(\lambda^s),$$

$$\nabla^2\prod_{p=1}^{s}(l_px + m_py + n_pz) = r^{s-2}\,.\,2\Sigma(\lambda^{s-2}\mu),$$

$$\nabla^4\prod_{p=1}^{s}(l_px + m_py + n_pz) = r^{s-4}\,.\,2^2\,.\,2\,!\ \Sigma(\lambda^{s-4}\mu^2),$$

since in $\Sigma(\lambda^{s-4}\mu^2)$ products such as $\mu_{12}\cdot\mu_{34}$ may be obtained in 2 ! ways,

$$\nabla^6\prod_{p=1}^{s}(l_px + m_py + n_pz) = r^{s-6}\,.\,2^3\,.\,3\,!\ \Sigma(\lambda^{s-6}\mu^3),$$

since the product $\mu_{12}\cdot\mu_{34}\cdot\mu_{56}$ may arise in 3 ! ways.

Proceeding in this way we obtain

$$\nabla^{2q}\prod_{p=1}^{s}(l_px + m_py + n_pz) = r^{s-2q}\,.\,2^q\,.\,q\,!\ \Sigma(\lambda^{s-2q}\mu^q), \qquad (12)$$

where $q = 1, 2, \ldots, s$.

Accordingly

$$Y_s = \frac{(2s)\,!}{2^s\,.\,(s\,!)^2}\left\{\Sigma(\lambda^s) - \frac{1}{2s-1}\Sigma(\lambda^{s-2}\mu) + \frac{1}{(2s-1)(2s-3)}\Sigma(\lambda^{s-4}\mu^2) - \ldots\right\}. \qquad (13)$$

Thus, for example, if P is the point (x, y, z) on the sphere of radius r, and A, B, C, D are the poles of the axes h_1, h_2, h_3, h_4,

$$Y_1 = \cos PA,$$

$$Y_2 = \tfrac{3}{2}(\cos PA\cos PB - \tfrac{1}{3}\cos AB),$$

$$Y_3 = \tfrac{5}{2}\{\cos PA\cos PB\cos PC - \tfrac{1}{5}(\cos PA\cos BC + \cos PB\cos CA + \cos PC\cos AB)\},$$

$$Y_4 = \frac{35}{8}\left\{\begin{array}{l}\cos PA\cos PB\cos PC\cos PD \\ -\dfrac{1}{7}\left(\begin{array}{l}\cos PA\cos PB\cos CD + \cos PB\cos PC\cos DA \\ +\cos PC\cos PD\cos AB + \cos PD\cos PA\cos BC \\ +\cos PA\cos PC\cos BD + \cos PB\cos PD\cos AC\end{array}\right) \\ +\frac{1}{35}(\cos AB\cos CD + \cos AC\cos BD + \cos AD\cos BC)\end{array}\right\}.$$

§ 3. Poles of Zonal, Tesseral, and Sectorial Harmonics. In (11) let all the poles be coincident on the z-axis; then

$$\frac{(-1)^n}{n!} r^{n+1} \frac{\partial^n}{\partial z^n} \frac{1}{r} = \frac{(2n)!}{2^n \cdot (n!)^2} \frac{1}{r^n}$$

$$\times \left[1 - \frac{r^2}{2(2n-1)} \frac{\partial^2}{\partial z^2} + \frac{r^4}{2 \cdot 4(2n-1)(2n-3)} \frac{\partial^4}{\partial z^4} - \dots \right] z^n$$

$$= \frac{(2n)!}{2^n \cdot (n!)^2} \left\{ \mu^n - \frac{n(n-1)}{2(2n-1)} \mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} \mu^{n-4} - \dots \right\}$$

$$= P_n(\mu), \text{ by (V., 6)}. \qquad (14)$$

Hence a zonal harmonic of integral degree n has all its n poles coincident on the z-axis. [For an alternative proof of (14) see *ex.* 9, p. 105.]

Again, let $n - m$ axes coincide with the z-axis, and let the remaining m axes lie in the (x, y) plane and make angles $\alpha, \alpha + \pi/m, \dots, \alpha + (m-1)\pi/m$ with the x-axis; if these m axes are $h_1, h_2, \dots h_m$,

$$\frac{\partial^m}{\partial h_1 \partial h_2 \dots \partial h_m} = \prod_{r=0}^{m-1} \left\{ \cos\left(\alpha + \frac{r\pi}{m}\right) \frac{\partial}{\partial x} + \sin\left(\alpha + \frac{r\pi}{m}\right) \frac{\partial}{\partial y} \right\}$$

$$= \frac{1}{2^m} \prod_{r=0}^{m-1} \left\{ e^{i\left(\alpha + \frac{r\pi}{m}\right)} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + e^{-i\left(\alpha + \frac{r\pi}{m}\right)} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right\}.$$

Here let $\xi = x + iy$, $\eta = x - iy$,
so that

$$2\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y};$$

then

$$\frac{\partial^m}{\partial h_1 \partial h_2 \dots \partial h_m} = \prod_{r=0}^{m-1} \left\{ e^{i\left(\alpha + \frac{r\pi}{m}\right)} \frac{\partial}{\partial \xi} + e^{-i\left(\alpha + \frac{r\pi}{m}\right)} \frac{\partial}{\partial \eta} \right\}.$$

Now the equation

$$x^m = y^m e^{-2im\alpha}$$

has solutions

$$x = y e^{-2i(\alpha + r\pi/m)}, \quad r = 0, 1, 2, \dots, m-1;$$

therefore

$$x^m - y^m e^{-2\iota m\alpha} = \prod_{r=0}^{m-1} \{x - ye^{-2\iota(\alpha + r\pi/m)}\},$$

or

$$e^{\iota m\alpha}x^m - e^{-\iota m\alpha}y^m = e^{-(m-1)\frac{1}{2}\iota\pi} \prod_{r=0}^{m-1} \left\{xe^{\iota\left(\alpha + \frac{r\pi}{m}\right)} - ye^{-\iota\left(\alpha + \frac{r\pi}{m}\right)}\right\}.$$

Accordingly

$$\frac{\partial^m}{\partial h_1 \partial h_2 \ldots \partial h_m} = e^{(m-1)\frac{1}{2}\iota\pi}\left\{e^{\iota m\alpha}\left(\frac{\partial}{\partial\xi}\right)^m - e^{-\iota m\alpha}\left(-\frac{\partial}{\partial\eta}\right)^m\right\}. \quad (15)$$

But, by (6),

$$\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^m \frac{1}{r} = (-1)^n \frac{(2n)!}{2^n . n!} \frac{1}{r^{2n+1}}$$

$$\times \left[1 - \frac{r^2}{2(2n-1)}\nabla^2 + \frac{r^4}{2\cdot 4(2n-1)(2n-3)}\nabla^4 - \ldots\right]$$

$$\times z^{n-m}(x \pm iy)^m$$

$$= (-1)^n \frac{(2n)!}{2^n . n!} \frac{1}{r^{2n+1}}(x \pm iy)^m$$

$$\times \left[z^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}z^{n-m-2}r^2 + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 . 4(2n-1)(2n-3)} \times z^{n-m-4}r^4 - \ldots\right]$$

$$= (-1)^n \frac{(2n)!}{2^n . n!} \frac{1}{r^{n+1}}(\cos m\phi \pm i \sin m\phi) \sin^m\theta$$

$$\times \left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 . 4(2n-1)(2n-3)} \times \mu^{n-m-4} - \ldots\right\}$$

$$= (-1)^n \frac{(n-m)!}{r^{n+1}}(\cos m\phi \pm i \sin m\phi) \mathrm{T}_n^m(\mu)$$

[Cf. Ch. VII., § 3, *ex.*].

Therefore

$$\frac{\partial^n}{\partial z^{n-m}\partial\xi^m}\,\frac{1}{r} = \frac{(-1)^n}{2^m}\,\frac{(n-m)!}{r^{n+1}}\,e^{-im\phi}\mathrm{T}_n^m(\mu), \quad . \quad (16)$$

and

$$\frac{\partial^n}{\partial z^{n-m}\partial\eta^m}\,\frac{1}{r} = \frac{(-1)^n}{2^m}\,\frac{(n-m)!}{r^{n+1}}\,e^{im\phi}\mathrm{T}_n^m(\mu). \quad . \quad (17)$$

Two cases will now be considered.

Case I.—In (15) assume that $\alpha = 0$; then

$$\frac{\partial^m}{\partial h_1\partial h_2\ .\ .\ .\ \partial h_m}\,\frac{1}{r} = e^{(m-1)\frac{1}{2}i\pi}\left\{\left(\frac{\partial}{\partial\xi}\right)^m - e^{-im\pi}\left(\frac{\partial}{\partial\eta}\right)^m\right\}\frac{1}{r},$$

so that, from (16) and (17),

$$\frac{\partial^n}{\partial z^{n-m}\partial h_1\partial h_2\ldots\partial h_m}\,\frac{1}{r} = \frac{(-1)^n}{2^m}\,\frac{(n-m)!}{r^{n+1}}\,2\sin m\left(\frac{\pi}{2}-\phi\right)\mathrm{T}_n^m(\mu). \quad (18)$$

Case II.—In (15) assume that $\alpha = \pi/(2m)$; then

$$\frac{\partial^m}{\partial h_1\partial h_2\ .\ .\ .\ \partial h_m}\,\frac{1}{r} = ie^{(m-1)\frac{1}{2}i\pi}\left\{\left(\frac{\partial}{\partial\xi}\right)^m + e^{-im\pi}\left(\frac{\partial}{\partial\eta}\right)^m\right\}\frac{1}{r},$$

so that

$$\frac{\partial^n}{\partial z^{n-m}\partial h_1\partial h_2\ldots\partial h_m}\,\frac{1}{r} = \frac{(-1)^n}{2^m}\,\frac{(n-m)!}{r^{n+1}}\,2\cos m\left(\frac{\pi}{2}-\phi\right)\mathrm{T}_n^m(\mu). \quad (19)$$

If m is even, the harmonics $\sin m\phi\mathrm{T}_n^m(\mu)$, $\cos m\phi\mathrm{T}_n^m(\mu)$ are given by (18) and (19) respectively, while if m is odd, they are given by (19) and (18) respectively. If the factors $\sin m(\frac{1}{2}\pi-\phi)$, $\cos m(\frac{1}{2}\pi-\phi)$ are equal to $-\sin m\phi$ or $-\cos m\phi$, the direction of one of the axes $h_1, h_2, \ .\ .\ .\ , h_m$ should be reversed. With this restriction the poles of the harmonics are those of the axes $h_1, h_2, \ .\ .\ .\ , h_m$, and $n-m$ coincident poles on the positive z-axis.

CHAPTER XIV

BESSEL FUNCTIONS

§ 1. **Bessel's Equation.** If Laplace's Equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

be transformed to cylindrical co-ordinates (u, ϕ, z) by means of the equations

$$x = u\cos\phi,\ y = u\sin\phi,\ u^2 = x^2 + y^2,\ \tan\phi = y/x,$$

it becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad (1)$$

In this equation put $V = U\Phi Z$, where U, Φ and Z are respectively functions of u, ϕ, and z alone, and divide by $U\Phi Z$; then

$$\frac{1}{U}\left(\frac{d^2 U}{du^2} + \frac{1}{u}\frac{dU}{du}\right) + \frac{1}{u^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \quad . \quad (2)$$

Here the first three terms are independent of z, and consequently so also is the last; this term has therefore a constant value C, and

$$\frac{d^2 Z}{dz^2} = CZ \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, since the first two and the last terms are independent of ϕ, so must also be the third; hence

$$\frac{d^2\Phi}{d\phi^2} = D\Phi, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where D is a constant. Thus (2) can be written

$$u^2\frac{d^2 U}{du^2} + u\frac{dU}{du} + (D + Cu^2)U = 0 \quad . \quad . \quad (5)$$

If now in these equations we put $C = \kappa^2$ and $D = -n^2$ (3) and (4) become

$$\frac{d^2Z}{dz^2} = \kappa^2 Z, \qquad (6)$$

and

$$\frac{d^2\Phi}{d\phi^2} = -n^2\Phi, \qquad (7)$$

of which the general solutions are

$$Z = Ae^{\kappa z} + Be^{-\kappa z}, \qquad (8)$$

and

$$\Phi = E\cos n\phi + F\sin n\phi; \qquad (9)$$

while (5) takes the form

$$u^2\frac{d^2U}{du^2} + u\frac{dU}{du} + (\kappa^2u^2 - n^2)U = 0,$$

or, if $v = \kappa u$,

$$v^2\frac{d^2U}{dv^2} + v\frac{dU}{dv} + (v^2 - n^2)U = 0. \qquad (10)$$

This is known as *Bessel's Differential Equation*, and the solutions of it are called *Cylinder Functions* or *Bessel Functions of order* n.

If $R_n(v)$ is a Bessel Function of order n, the function

$$V = R_n(\kappa u)\,{}^{\sin}_{\cos}(n\phi)e^{\pm\kappa z} \qquad (11)$$

is a solution of Laplace's Equation. A function of this kind is called a *Cylindrical Harmonic*. If $n = 0$, the Harmonic is symmetrical about the z-axis.

§ 2. **Solution of the Differential Equation.** To solve Bessel's Equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad (12)$$

put

$$y = x^\rho\sum_{r=0}^{\infty} c_rx^r. \qquad (13)$$

in the left-hand side of the equation, and it becomes

$$x^\rho\sum_{r=0}^{\infty} c_r(\rho + r)(\rho + r - 1)x^r + x^\rho\sum_{r=0}^{\infty} c_r(\rho + r)x^r$$
$$+ (x^2 - n^2)x^\rho\sum_{r=0}^{\infty} c_rx^r$$

or
$$x^\rho c_0(\rho^2 - n^2) + x^{\rho+1}c_1\{(\rho+1)^2 - n^2\}$$
$$+ x^\rho \sum_{r=2}^{\infty} [c_r\{(\rho+r)^2 - n^2\} + c_{r-2}]x^r. \quad (14)$$

In order that the series (13) should satisfy (12), it is necessary that the expression (14) should vanish identically; that is, that the coefficients of all the powers of x in (14) should be zero. Thus $c_0(\rho^2 - n^2)$, the coefficient of x^ρ, must vanish, and as c_0 is the coefficient of the first term in the series, it cannot be zero; hence
$$\rho^2 - n^2 = 0 \quad . \quad . \quad . \quad . \quad (15)$$

This equation is known as the *Indicial Equation*, since the values of ρ given by it determine the index ρ. These two values, of course, are $\pm n$.

If now the coefficients of $x^{\rho+1}$, $x^{\rho+2}$, . . . in (14) are equated to zero, the equations

$$\begin{array}{lcl} c_1\{(\rho+1)^2 - n^2\} = 0 & & c_1(\rho+n+1)(\rho-n+1) = 0 \\ c_2\{(\rho+2)^2 - n^2\} + c_0 = 0 & & c_2(\rho+n+2)(\rho-n+2) = -c_0 \\ . \quad . \quad . \quad . \quad . \quad . & \text{or} & . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16) \\ c_r\{(\rho+r)^2 - n^2\} + c_{r-2} = 0 & & c_r(\rho+n+r)(\rho-n+r) = -c_{r-2} \\ . \quad . \quad . \quad . \quad . \quad . & & . \quad . \quad . \quad . \quad . \quad . \end{array}$$

are obtained, and from them the coefficients c_1, c_2, c_3, . . . can be found in terms of c_0. From the first equation it appears that $c_1 = 0$, and consequently c_3, c_5, . . . and all the coefficients with odd suffixes are zero. From the remaining equations it follows that

$$\begin{aligned} c_2 &= \frac{-c_0}{(\rho-n+2)(\rho+n+2)}, \\ c_4 &= \frac{c_0}{(\rho-n+2)(\rho-n+4)(\rho+n+2)(\rho+n+4)}, \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17) \\ c_{2r} &= \frac{(-1)^r c_0}{\left[\begin{array}{l}(\rho-n+2)(\rho-n+4)\ldots(\rho-n+2r) \\ \quad \times (\rho+n+2)(\rho+n+4)\ldots(\rho+n+2r)\end{array}\right]} \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

When the coefficients are given in terms of c_0 by (17), the equations (16) are satisfied, and the expression (14) reduces to $x^\rho c_0(\rho^2 - n^2)$, so that the function (13) now satisfies the equation

$$x^2y'' + xy' + (x^2 - n^2)y = x^\rho c_0(\rho^2 - n^2). \quad . \quad (18)$$

If now in (17) and (13) we put $\rho = \pm n$, the function (13) is a solution of (18) with $\rho = \pm n$, that is of Bessel's Equation (12), provided that the series in (13) is convergent.

For instance, if $\rho = n$, the solution obtained is

$$y = c_0x^n\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \dots\right\},$$

provided that n is not a negative integer. If we here put

$$c_0 = \frac{1}{2^n\Gamma(n+1)},$$

we obtain the function

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \dots\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which is known as the *Bessel Function of the First Kind of order* n. The series on the right of (19) is absolutely convergent for all values of x.

If n is not an integer, a second solution is obtained by putting $\rho = -n$ in (17), (13), and (18); it is

$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(-n+1)}$$

$$\times \left\{1 - \frac{x^2}{2(-2n+2)} + \frac{x^4}{2 \cdot 4(-2n+2)(-2n+4)} - \dots\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}. \quad . \quad . \quad . \quad (20)$$

Thus, if n is not an integer, the general solution of Bessel's Equation is

$$y = AJ_n(x) + BJ_{-n}(x),$$

where A and B are arbitrary constants.

On the other hand, if n is zero or an integer, these two solutions are not distinct. If n is zero, they are obviously identical; while, if n is a positive integer, since $1/\Gamma(-p)$ is zero if p is zero or a positive integer, the first n coefficients in (20) vanish, and

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r};$$

or, if $r = n + s$,

$$\begin{aligned} J_{-n}(x) &= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s} \\ &= (-1)^n J_n(x) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (21) \end{aligned}$$

To solve the equation completely it is therefore necessary, in these cases, to find a second solution.

Case I. : n = 0.—When $n = 0$, equation (18) becomes

$$x^2y'' + xy' + x^2y = x^\rho c_0\rho^2, \qquad . \quad . \quad . \quad (22)$$

of which, from (13) and (17), a solution is

$$y = c_0x^\rho\left\{1 - \frac{x^2}{(\rho+2)^2} + \frac{x^4}{(\rho+2)^2(\rho+4)^2} - \dots\right\}. \quad (23)$$

Here put $\rho = 0$, and (22) reduces to Bessel's Equation with $n = 0$, and, from (23), one solution of this equation is

$$y = c_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \dots\right) = c_0J_0(x).$$

Now differentiate (22) with regard to ρ, and it becomes

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial y}{\partial \rho}\right) + x\frac{d}{dx}\left(\frac{\partial y}{\partial \rho}\right) + x^2\left(\frac{\partial y}{\partial \rho}\right) = x^\rho c_0\rho(2 + \rho\log x), \quad (24)$$

of which, from (23), a solution is

$$\begin{aligned} \left(\frac{\partial y}{\partial \rho}\right) = {} & c_0x^\rho \log x\left\{1 - \frac{x^2}{(\rho+2)^2} + \frac{x^4}{(\rho+2)^2(\rho+4)^2} - \dots\right\} \\ & + c_0x^\rho\left\{\frac{x^2}{(\rho+2)^2}\,\frac{2}{\rho+2} - \frac{x^4}{(\rho+2)^2(\rho+4)^2}\left(\frac{2}{\rho+2} + \frac{2}{\rho+4}\right) + \dots\right\}. \end{aligned}$$

$$(25)$$

But, if $\rho = 0$, (24) reduces to Bessel's Equation with $n = 0$, of which (25) gives the solution

$$\left(\frac{\partial y}{\partial \rho}\right)_{\rho=0} = c_0Y_0(x),$$

where

$$Y_0(x) = J_0(x)\log x + \frac{x^2}{2^2} - \frac{x^4}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \ldots ; \quad (26)$$

this function is known as *Neumann's Bessel Function of the Second Kind of order zero.*

Case II. : n *a positive integer.*—In (18) put $c_0 = c(\rho + n)$, and it becomes

$$x^2y'' + xy' + (x^2 - n^2)y = x^\rho c(\rho - n)(\rho + n)^2 . \quad (27)$$

Of this, from (13) and (17), a solution is

$$\begin{aligned} y &= c(\rho + n)x^\rho\left\{1 - \frac{x^2}{(\rho - n + 2)(\rho + n + 2)}\right. \\ &\qquad \left. + \frac{x^4}{(\rho - n + 2)(\rho - n + 4)(\rho + n + 2)(\rho + n + 4)} - \ldots\right\} \\ &= c(\rho + n)x^\rho\left\{1 - \frac{x^2}{(\rho - n + 2)(\rho + n + 2)} + \ldots\right. \\ &\qquad \left. + \frac{(-1)^{n-1}x^{2n-2}}{(\rho - n + 2)\ldots(\rho + n - 2)(\rho + n + 2)\ldots(\rho + 3n - 2)}\right\} \\ &+ \frac{(-1)^n c x^{\rho + 2n}}{(\rho - n + 2)\ldots(\rho + n - 2)(\rho + n + 2)\ldots(\rho + 3n)} \\ &\times\left\{1 - \frac{x^2}{(\rho + n + 2)(\rho + 3n + 2)}\right. \\ &\qquad \left. + \frac{x^4}{(\rho + n + 2)(\rho + n + 4)(\rho + 3n + 2)(\rho + 3n + 4)} - \ldots\right\}. \end{aligned} \quad (28)$$

Here put $\rho = -n$, and (27) reduces to Bessel's Equation, while, from (28), we obtain the solution

$$y = \frac{-c}{2^{n-1}(n - 1)!}J_n(x).$$

Again, differentiating (27) with regard to ρ, we obtain the equation

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial y}{\partial \rho}\right) + x\frac{d}{dx}\left(\frac{\partial y}{\partial \rho}\right) + (x^2 - n^2)\left(\frac{\partial y}{\partial \rho}\right) = x^\rho c(\rho + n)\{3\rho - n + (\rho^2 - n^2)\log x\},$$

of which, from (28) a solution is

$$\begin{aligned}
\frac{\partial y}{\partial \rho} = y \log x &+ cx^{\rho}\left\{1 - \frac{x^2}{(\rho - n + 2)(\rho + n + 2)} + \ldots \right. \\
&\left. + \frac{(-1)^{n-1}x^{2n-2}}{(\rho - n + 2)\ldots(\rho + n - 2)(\rho + n + 2)\ldots(\rho + 3n - 2)}\right\} \\
&+ (\rho + n)[\ldots] \\
&+ \frac{(-1)^{n-1}cx^{\rho+2n}}{(\rho - n + 2)\ldots(\rho + n - 2)(\rho + n + 2)\ldots(\rho + 3n)} \\
&\times\left\{\begin{array}{l} \dfrac{1}{\rho - n + 2} + \dfrac{1}{\rho - n + 4} + \ldots + \dfrac{1}{\rho + n - 2} \\ + \dfrac{1}{\rho + n + 2} + \dfrac{1}{\rho + n + 4} + \ldots + \dfrac{1}{\rho + 3n} \end{array}\right\} \\
&\times\left\{\begin{array}{l} 1 - \dfrac{x^2}{(\rho + n + 2)(\rho + 3n + 2)} \\ + \dfrac{x^4}{(\rho + n + 2)(\rho + n + 4)(\rho + 3n + 2)(\rho + 3n + 4)} - \ldots \end{array}\right\} \\
&+ \frac{(-1)^{n}cx^{\rho+2n}}{(\rho - n + 2)\ldots(\rho + n - 2)(\rho + n + 2)\ldots(\rho + 3n)} \\
&\times\left\{\begin{array}{l} \dfrac{x^2}{(\rho + n + 2)(\rho + 3n + 2)}\left(\dfrac{1}{\rho + n + 2} + \dfrac{1}{\rho + 3n + 2}\right) \\ - \dfrac{x^4}{(\rho + n + 2)(\rho + n + 4)(\rho + 3n + 2)(\rho + 3n + 4)} \\ \times\left(\dfrac{1}{\rho + n + 2} + \dfrac{1}{\rho + n + 4} + \dfrac{1}{\rho + 3n + 2} + \dfrac{1}{\rho + 3n + 4}\right) + \ldots \end{array}\right\}.
\end{aligned}$$

If we now put $\rho = -n$, the equation becomes Bessel's Equation, and the solution is

$$\begin{aligned}
\left(\frac{\partial y}{\partial \rho}\right)_{\rho = -n} &= \frac{-c}{2^{n-1}(n-1)!}\,\mathrm{J}_n(x)\log x \\
&\quad + cx^{-n}\left\{\begin{array}{l} 1 + \dfrac{x^2}{2(2n-2)} + \dfrac{x^4}{2\ 4(2n-2)(2n-4)} + \ldots \\ + \dfrac{x^{2n-2}}{2\cdot 4\ldots(2n-2)(2n-2)\ldots 4\cdot 2} \end{array}\right\} \\
&+ \frac{c}{2^n n!}\,\mathrm{J}_n(x) \\
&- \frac{cx^n}{2^{2n-1}(n-1)!\ n!}\left\{\frac{x^2}{2(2n+2)}\left(\frac{1}{2} + \frac{1}{2n+2}\right)\right. \\
&\qquad - \frac{x^4}{2\,.\,4(2n+2)(2n+4)} \\
&\qquad \left.\times\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2n+2} + \frac{1}{2n+4}\right) + \ldots\right\}.
\end{aligned}$$

Here let $c = -2^{n-1}(n-1)!$, and the solution of Bessel's Equation is

$$y = J_n(x)\log x - 2^{n-1}(n-1)!\,x^{-n}\left\{1 + \frac{x^2}{2(2n-2)} + \ldots \right.$$
$$\left. \ldots + \frac{x^{2n-2}}{2 \cdot 4 \ldots (2n-2)(2n-2)\ldots 4 \cdot 2}\right\}$$
$$- \frac{1}{2n}J_n(x) + \frac{x^n}{2^n n!}\left\{\frac{x^2}{2(2n+2)}\left(\frac{1}{2} + \frac{1}{2n+2}\right)\right.$$
$$\left. - \frac{x^4}{2 \cdot 4(2n+2)(2n+4)}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2n+2} + \frac{1}{2n+4}\right) + \ldots\right\}.$$

From this subtract $\frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1}\right)J_n(x)$, and the difference is the function

$$Y_n(x) = J_n(x)\log x - 2^{n-1}(n-1)!\,x^{-n}\left\{1 + \frac{x^2}{2(2n-2)} + \ldots \right.$$
$$\left. \ldots + \frac{x^{2n-2}}{2 \cdot 4 \ldots (2n-2)(2n-2)\ldots 4 \cdot 2}\right\}$$
$$- \frac{1}{2}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\{\phi(r) + \phi(n+r)\} \quad . \quad (29)$$

where $\phi(r) = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{r}$ and $\phi(0) = 0$.

$Y_n(x)$ is called *Neumann's Bessel Function of the Second Kind of order* n.

Bessel Functions of the second kind have been defined in various other ways by different writers. All such functions, of course, must be expressible in the form

$$AJ_n(x) + BY_n(x).$$

The definition of a function of this kind, denoted by $G_n(x)$, will be found in § 8.

§ 3. **Recurrence Formulæ for $J_n(x)$.** When the argument x of the function $J_n(x)$ remains the same throughout, it is frequently convenient to write J_n for $J_n(x)$, and J'_n for $\frac{d}{dx}J_n(x)$.

Now, from (19),

$$xJ'_n = \sum_{r=0}^{\infty} \frac{(-1)^r(n+2r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \quad . \quad . \quad . \quad (30)$$

$$= nJ_n + x\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n - x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+s+2)}\left(\frac{x}{2}\right)^{n+2s+1}$$

$$= nJ_n - xJ_{n+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Again, from (30),

$$xJ'_n = -nJ_n + x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n + xJ_{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Adding (31) and (32), and dividing by x, we get

$$2J'_n = J_{n-1} - J_{n+1}, \quad . \quad . \quad . \quad (33)$$

and, in particular, when $n = 0$, since $J_{-1} = -J_1$,

$$J'_0 = -J_1 \quad . \quad . \quad . \quad . \quad (34)$$

Again, subtracting (32) from (31), we find that

$$\frac{2n}{x}J_n = J_{n-1} + J_{n+1}. \quad . \quad . \quad . \quad (35)$$

If (32) and (31) are multiplied by x^{n-1} and x^{-n-1}, respectively, they can be written

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}, \quad . \quad . \quad . \quad (36)$$

and

$$\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1} \quad . \quad . \quad (37)$$

Finally, by differentiating (33), we get

$$\begin{aligned} 2^2 J''_n &= 2J'_{n-1} - 2J'_{n+1} \\ &= (J_{n-2} - J_n) - (J_n - J_{n+2}) \\ &= J_{n-2} - 2J_n + J_{n+2}, \end{aligned}$$

and, by induction, it can be shown that

$$2^r J_n^{(r)} = J_{n-r} - rJ_{n-r+2} + \frac{r(r-1)}{2!}J_{n-r+4} - \dots + (-1)^r J_{n+r} \tag{38}$$

§ 4. **Expressions for $J_n(x)$ when n is half an odd integer.** When $n = \frac{1}{2}$, it follows from (19) that

$$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{3}{2})}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x; \tag{39}$$

while, when $n = -\frac{1}{2}$,

$$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\Gamma(\frac{1}{2})}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x. \tag{40}$$

Hence, and by means of (35), it is possible to express $J_{k+\frac{1}{2}}(x)$, where k is an integer, as a sum of a finite number of terms involving $\sin x$, $\cos x$, and powers of x. The formula for *$J_n(x)$ in this case is given in (XV., 57).*

§ 5. **The Bessel Coefficients.** When t is not zero, the functions $e^{\frac{1}{2}xt}$ and $e^{-\frac{1}{2}x/t}$ can be expanded in powers of t, and the product of these expansions is

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xt}{2}\right)^r\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{x}{2t}\right)^s$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{r!\,s!}\left(\frac{x}{2}\right)^{r+s}t^{r-s}.$$

If n is zero or a positive integer, the coefficient of t^n in this expansion is obtained by making s vary from 0 to infinity, while r takes the values given by $r = n + s$; thus its value is

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s} = J_n(x).$$

If n is a negative integer, the coefficient of t^n is obtained by making r vary from 0 to infinity, while s takes the values given by $s = -n + r$; its value is therefore

$$\sum_{r=0}^{\infty}\frac{(-1)^{-n+r}}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r} = (-1)^{-n}J_{-n}(x) = J_n(x).$$

Hence

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty} t^n \mathrm{J}_n(x). \quad . \quad . \quad . \quad (41)$$

Because of their appearance as coefficients in this expansion, the functions J_n of integral order are known as *Bessel Coefficients.*

From (41) many properties of the Bessel Coefficients can be derived. For example, put $t = e^{i\phi}$, and (41) becomes

$$\begin{aligned} e^{ix\sin\phi} = \mathrm{J}_0(x) &+ 2i\sin\phi \mathrm{J}_1(x) + 2\cos 2\phi \mathrm{J}_2(x) \\ &+ 2i\sin 3\phi \mathrm{J}_3(x) + 2\cos 4\phi \mathrm{J}_4(x) + \ldots \quad (42) \end{aligned}$$

Equating the real and imaginary parts of this identity, we deduce that

$$\cos(x\sin\phi) = \mathrm{J}_0(x) + 2\cos 2\phi \mathrm{J}_2(x) + 2\cos 4\phi \mathrm{J}_4(x) + \ldots, \quad (43)$$

and

$$\begin{aligned} \sin(x\sin\phi) = 2\sin\phi \mathrm{J}_1(x) &+ 2\sin 3\phi \mathrm{J}_3(x) \\ &+ 2\sin 5\phi \mathrm{J}_5(x) + \ldots \quad (44) \end{aligned}$$

In the identities (42), (43), and (44) replace ϕ by $\frac{1}{2}\pi - \phi$, and they become

$$\begin{aligned} e^{ix\cos\phi} = \mathrm{J}_0(x) &+ 2i\cos\phi \mathrm{J}_1(x) + 2i^2\cos 2\phi \mathrm{J}_2(x) \\ &+ 2i^3\cos 3\phi \mathrm{J}_3(x) + \ldots, \quad (45) \end{aligned}$$

$$\cos(x\cos\phi) = \mathrm{J}_0(x) - 2\cos 2\phi \mathrm{J}_2(x) + 2\cos 4\phi \mathrm{J}_4(x) - \ldots, \quad (46)$$

and

$$\begin{aligned} \sin(x\cos\phi) = 2\cos\phi \mathrm{J}_1(x) &- 2\cos 3\phi \mathrm{J}_3(x) \\ &+ 2\cos 5\phi \mathrm{J}_5(x) - \ldots \quad (47) \end{aligned}$$

Bessel's Integral. Multiply (43) by $\cos n\phi$, and integrate from 0 to π; then

$$\begin{aligned} \int_0^\pi \cos(x\sin\phi)\cos n\phi \, d\phi &= \pi \mathrm{J}_n(x), \text{ if } n \text{ is zero or an even integer,} \\ &= 0, \text{ if } n \text{ is odd.} \end{aligned}$$

Similarly, from (44),

$$\begin{aligned} \int_0^\pi \sin(x\sin\phi)\sin n\phi \, d\phi &= 0, \text{ if } n \text{ is zero or an even integer,} \\ &= \pi \mathrm{J}_n(x), \text{ if } n \text{ is odd.} \end{aligned}$$

Hence, on addition,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)d\phi, \quad . \quad . \quad (48)$$

where n is zero or a positive integer. It was in this form that the function $J_n(x)$ was first obtained by Bessel.

The Addition Theorem. From (41)

$$\sum_{n=-\infty}^{\infty} t^n J_n(u+v) = e^{\frac{1}{2}(u+v)\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}u\left(t-\frac{1}{t}\right)}e^{\frac{1}{2}v\left(t-\frac{1}{t}\right)}$$

$$= \sum_{r=-\infty}^{\infty} t^r J_r(u) \sum_{s=-\infty}^{\infty} t^s J_s(v). \quad . \quad (49)$$

Here equate the coefficients of t^n; the terms in t^n on the right are obtained by making r vary from $-\infty$ to $+\infty$, and taking $s = n - r$; thus, if n is any integer

$$J_n(u+v) = \sum_{r=-\infty}^{\infty} J_r(u)J_{n-r}(v). \quad . \quad . \quad (50)$$

When n is a positive integer, this can be written

$$J_n(u+v) = \sum_{r=0}^{n} J_r(u)J_{n-r}(v)$$

$$+ \sum_{r=1}^{\infty} (-1)^r\{J_r(u)J_{n+r}(v) + J_r(v)J_{n+r}(u)\}. \quad (51)$$

Neumann's Addition Formula for J_0. When $n = 0$, a more general form of the addition theorem can be obtained as follows. Since

$$e^{\frac{x}{2}\left(kt-\frac{1}{kt}\right)} = e^{\frac{x}{2t}\left(k-\frac{1}{k}\right)}e^{\frac{kx}{2}\left(t-\frac{1}{t}\right)},$$

it can be deduced from (41) that

$$\sum_{n=-\infty}^{\infty} J_n(x)k^n t^n = e^{\frac{x}{2t}\left(k-\frac{1}{k}\right)} \sum_{n=-\infty}^{\infty} J_n(kx)t^n.$$

Here replace x and k by b and $e^{i\beta}$ and then by c and $e^{-i\gamma}$, and multiply the resulting equations together; thus

$$\sum_{m=-\infty}^{\infty} e^{im\beta}J_m(b)t^m \sum_{n=-\infty}^{\infty} e^{-in\gamma}J_n(c)t^n$$

$$= e^{\frac{i}{t}(b\sin\beta - c\sin\gamma)} \sum_{m=-\infty}^{\infty} J_m(be^{i\beta})t^m \sum_{n=-\infty}^{\infty} J_n(ce^{-i\gamma})t^n$$

$$= e^{\frac{i}{t}(b\sin\beta - c\sin\gamma)} \sum_{n=-\infty}^{\infty} J_n(be^{i\beta} + ce^{-i\gamma})t^n, \quad . \quad . \quad (52)$$

by (49).

Here let $a = be^{i\beta} + ce^{-i\gamma}$, and let β and γ be chosen so that $b\sin\beta - c\sin\gamma = 0$, and consequently $a = b\cos\beta + c\cos\gamma$. Then, if a, b and c are the sides of a triangle, γ and β may be taken to be the angles opposite b and c, and, if α is the remaining angle, $\alpha = \pi - \beta - \gamma$, and

$$\begin{aligned} a^2 &= (b\cos\beta + c\cos\gamma)^2 + (b\sin\beta - c\sin\gamma)^2 \\ &= b^2 + c^2 + 2bc\cos(\beta + \gamma) = b^2 + c^2 - 2bc\cos\alpha. \end{aligned}$$

Hence, equating the terms independent of t in (52), we deduce that

$$J_0\{\sqrt{(b^2 + c^2 - 2bc\cos\alpha)}\} = \sum_{n=-\infty}^{\infty} e^{in(\pi-\alpha)}J_n(b)J_{-n}(c)$$

$$= J_0(b)J_0(c) + 2\sum_{n=1}^{\infty} J_n(b)J_n(c)\cos n\alpha. \quad . \quad (53)$$

§ 6. **Expansion of x^n in a Series of Bessel Functions.** If n is zero, or a positive integer, then *

$$2\cos n\phi = (2\cos\phi)^n - \frac{n}{1}(2\cos\phi)^{n-2} + \frac{n(n-3)}{2!}(2\cos\phi)^{n-4} - \dots$$

$$+ (-1)^r\frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!} \times (2\cos\phi)^{n-2r} + \dots$$

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* This formula can be deduced from the identity

$$\frac{1-x^2}{1-2x\cos\phi + x^2} = 1 + 2x\cos\phi + 2x^2\cos 2\phi + 2x^3\cos 3\phi + \dots$$

by expanding the expression on the left in powers of x, and equating the coefficients of x^n.

If now this formula be applied to the terms on the right of the *identity* (45), and the coefficients of $(\cos\phi)^n$ equated, it is found that

$$x^n = 2^n n! \left\{ J_n + \frac{n+2}{1} J_{n+2} + \frac{(n+4)(n+1)}{2!} J_{n+4} + \frac{(n+6)(n+2)(n+1)}{3!} J_{n+6} + \ldots \right\}$$

$$= 2^n \sum_{r=0}^{\infty} \frac{(n+2r)(n+r-1)!}{r!} J_{n+2r}, \qquad (54)$$

where n is zero or a positive integer.

In particular, when $n = 0, 1, 2,$

$$1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots,$$
$$x = 2J_1 + 6J_3 + 10J_5 + 14J_7 + \ldots,$$
$$x^2 = 2(4J_2 + 16J_4 + 36J_6 + \ldots).$$

The formula (54) may be written

$$x^n = 2^n \sum_{r=0}^{\infty} \frac{(n+2r)\Gamma(n+r)}{r!} J_{n+2r}, \qquad (55)$$

and it will now be shown that this formula holds for all values of n except negative integers. Let

$$f(x) = \left(\frac{2}{x}\right)^n \sum_{r=0}^{\infty} \frac{(n+2r)\Gamma(n+r)}{r!} J_{n+2r}. \qquad (55')$$

Then

$$f'(x) = -n\left(\frac{2}{x}\right)^n \sum_{r=0}^{\infty} \frac{\Gamma(n+r)}{r!} \frac{n+2r}{x} J_{n+2r} + \left(\frac{2}{x}\right)^n \sum_{r=0}^{\infty} \frac{(n+2r)\Gamma(n+r)}{r!} J'_{n+2r};$$

and, by means of (35) and (33), this can be written

$$f'(x) = \left(\frac{2}{x}\right)^n \sum_{r=0}^{\infty} \frac{\Gamma(n+r)}{r!} \left\{ -\frac{n}{2}(J_{n+2r-1} + J_{n+2r+1}) + \frac{n+2r}{2}(J_{n+2r-1} - J_{n+2r+1}) \right\}$$

$$= \left(\frac{2}{x}\right)^n \sum_{r=0}^{\infty} \frac{\Gamma(n+r)}{r!}\{rJ_{n+2r-1} - (n+r)J_{n+2r+1}\}$$

$$= \left(\frac{2}{x}\right)^n \left\{\sum_{r=1}^{\infty} \frac{\Gamma(n+r)}{(r-1)!}J_{n+2r-1} - \sum_{r=0}^{\infty} \frac{\Gamma(n+r+1)}{r!}J_{n+2r+1}\right\}.$$

The terms in these two summations cancel, and therefore $f'(x) = 0$. Thus $f(x)$ is constant, and, by making x tend to zero in (55′), we see that this constant value is unity. Hence, multiplying by x^n, we obtain the identity (55).

§ 7. **Definite Integral Expressions for the Bessel Functions.** In Bessel's Equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

put $y = x^n z$, and it becomes

$$xz'' + (2n + 1)z' + xz = 0 \quad . \quad . \quad (56)$$

To obtain a solution of this equation, we make the substitution

$$z = \int\phi(\lambda)e^{\lambda x}d\lambda,$$

and the equation takes the form

$$\int\phi(\lambda)e^{\lambda x}\{\lambda^2 x + (2n + 1)\lambda + x\}d\lambda = 0. \quad . \quad (57)$$

Hence, if $\phi(\lambda)$ satisfy the differential equation

$$\frac{d}{d\lambda}\{\phi(\lambda)(\lambda^2 + 1)\} = \phi(\lambda)(2n + 1)\lambda, \quad . \quad . \quad (58)$$

the left-hand side of (57) will become

$$\int\frac{d}{d\lambda}\{\theta(\lambda)\}d\lambda,$$

where $\theta(\lambda) = e^{\lambda x}\phi(\lambda)(\lambda^2 + 1)$, and equation (57) will be satisfied if $\theta(\lambda)$ has equal values at the limits of the integral.

Now (58) can be written

$$\frac{\frac{d}{d\lambda}\{\phi(\lambda)(\lambda^2 + 1)\}}{\phi(\lambda)(\lambda^2 + 1)} = \frac{(2n + 1)\lambda}{\lambda^2 + 1},$$

of which an integral is

$$\phi(\lambda)(\lambda^2 + 1) = C(\lambda^2 + 1)^{n+\frac{1}{2}},$$

where C is an arbitrary constant; thus

$$y = x^n\int e^{\lambda x}(\lambda^2 + 1)^{n-\frac{1}{2}}d\lambda \quad . \quad . \quad . \quad (59)$$

is a solution of Bessel's Equation, provided that

$$\theta(\lambda) = e^{\lambda x}(\lambda^2 + 1)^{n+\frac{1}{2}}$$

has equal values at the limits of integration.

If in (59) $i\lambda$ is put in place of λ, the solution becomes

$$y = x^n\int e^{i\lambda x}(1 - \lambda^2)^{n-\frac{1}{2}}d\lambda, \quad . \quad . \quad (60)$$

with

$$\theta(\lambda) = e^{i\lambda x}(1 - \lambda^2)^{n+\frac{1}{2}} \quad . \quad . \quad . \quad (61)$$

For example, if $n > -\frac{1}{2}$, the limits of integration may be taken to be ± 1, and

$$\mathrm{I} \equiv x^n\int_{-1}^{1} e^{i\lambda x}(1 - \lambda^2)^{n-\frac{1}{2}}d\lambda$$

is a Bessel Function. To find its value, expand $e^{i\lambda x}$ in powers of λ, and integrate term by term. The terms containing odd powers of λ will have odd integrands, and will therefore vanish; hence

$$\mathrm{I} = x^n\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2r)!}2\int_0^1\lambda^{2r}(1 - \lambda^2)^{n-\frac{1}{2}}d\lambda.$$

Here put $\lambda^2 = \mu$, and the integral becomes

$$\tfrac{1}{2}\int_0^1\mu^{r-\frac{1}{2}}(1 - \mu)^{n-\frac{1}{2}}d\mu = \tfrac{1}{2}\mathrm{B}(r + \tfrac{1}{2}, n + \tfrac{1}{2});$$

so that

$$\mathrm{I} = \frac{\Gamma(\frac{1}{2})\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)}x^n\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2r)!}\,\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}}{(n + 1)(n + 2)\ldots(n + r)}$$

$$= \sqrt{\pi}\,.\,\Gamma(n + \tfrac{1}{2})x^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n + r + 1)}\left(\frac{x}{2}\right)^{2r}$$

$$= \sqrt{\pi}\,.\,\Gamma(n + \tfrac{1}{2})2^n\mathrm{J}_n(x).$$

Thus

$$\mathrm{J}_n(x) = \frac{1}{\sqrt{\pi}\,.\,\Gamma(n + \frac{1}{2})}\left(\frac{x}{2}\right)^n\int_{-1}^{1}e^{i\lambda x}(1 - \lambda^2)^{n-\frac{1}{2}}d\lambda \quad (62)$$

In the same way it can be shown that

$$\mathrm{J}_n(x) = \frac{1}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_{-1}^{1} e^{-i\lambda x}(1-\lambda^2)^{n-\frac{1}{2}}d\lambda. \qquad (63)$$

On adding (62) and (63), we find that

$$\mathrm{J}_n(x) = \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_{0}^{1} \cos(\lambda x)(1-\lambda^2)^{n-\frac{1}{2}}d\lambda; \qquad (64)$$

and, on writing $\lambda = \sin\phi$, this becomes

$$\mathrm{J}_n(x) = \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_{0}^{\pi} \cos(x\sin\phi)(\cos\phi)^{2n}d\phi. \qquad (65)$$

Finally, on replacing ϕ by $\frac{1}{2}\pi - \phi$ in the last integral, we get

$$\mathrm{J}_n(x) = \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_{0}^{\frac{\pi}{2}} \cos(x\cos\phi)(\sin\phi)^{2n}d\phi. \qquad (66)$$

§ 8. **The Function $\mathrm{G}_n(x)$.** The function

$$\mathrm{G}_n(x) = \frac{\pi}{2\sin n\pi}\{\mathrm{J}_{-n}(x) - e^{-in\pi}\mathrm{J}_n(x)\} \quad . \quad (67)$$

may be employed in place of $\mathrm{Y}_n(x)$ as the Bessel Function of the second kind. When n tends to zero or an integer, the numerator and the denominator on the right-hand side of (67) both tend to zero. The function $\mathrm{G}_n(x)$ is then defined as the limit of the ratio, so that

$$\mathrm{G}_n = \frac{1}{2}\left[(-1)^n\frac{\partial}{\partial m}\mathrm{J}_{-m} - \frac{\partial}{\partial m}\mathrm{J}_m + i\pi\mathrm{J}_n\right]_{m=n}.$$

Now, from (19),

$$\mathrm{J}_m = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(m+r+1)}\left(\frac{x}{2}\right)^{m+2r},$$

so that

$$\frac{\partial}{\partial m}\mathrm{J}_m = \mathrm{J}_m\log\left(\frac{x}{2}\right) - \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(m+r+1)}\,\frac{\Gamma'(m+r+1)}{\Gamma(m+r+1)}\left(\frac{x}{2}\right)^{m+2r}.$$

Again, from (19) and (IV., 40)

$$\mathrm{J}_{-m} = \sum_{r=0}^{n-1}\frac{\Gamma(m-r)}{r!}\,\frac{\sin m\pi}{\pi}\left(\frac{x}{2}\right)^{-m+2r}$$

$$+ \sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-m+r+1)}\left(\frac{x}{2}\right)^{-m+2r},$$

and therefore

$$\frac{\partial}{\partial m}J_{-m} = -J_{-m}\log\left(\frac{x}{2}\right)$$
$$+\sum_{r=0}^{n-1}\frac{1}{r!}\left\{\Gamma'(m-r)\frac{\sin m\pi}{\pi}+\Gamma(m-r)\cos m\pi\right\}\left(\frac{x}{2}\right)^{-m+2r}$$
$$+\sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-m+r+1)}\left(\frac{x}{2}\right)^{-m+2r}\frac{\Gamma'(-m+r+1)}{\Gamma(-m+r+1)}.$$

Thus, using (IV., 48, 50), we get

$$G_n = \frac{1}{2}\left[\begin{aligned} & -J_n\log\left(\frac{x}{2}\right)+\sum_{r=0}^{n-1}\frac{\Gamma(n-r)}{r!}\left(\frac{x}{2}\right)^{-n+2r} \\ & \qquad +\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\{\phi(r)-\gamma\}\left(\frac{x}{2}\right)^{n+2r} \\ & -J_n\log\left(\frac{x}{2}\right)+\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\{\phi(n+r)-\gamma\}\left(\frac{x}{2}\right)^{n+2r} \\ & \qquad\qquad + i\pi J_n \end{aligned}\right]$$

$$= -J_n\log\left(\frac{x}{2}\right)+\frac{1}{2}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r}$$
$$+\frac{1}{2}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\{\phi(r)+\phi(n+r)-2\gamma\}\left(\frac{x}{2}\right)^{n+2r}+\tfrac{1}{2}i\pi J_n. \tag{68}$$

On comparing this with (29), we see that

$$G_n = -Y_n + J_n(\log 2 - \gamma + \tfrac{1}{2}i\pi), \tag{69}$$

so that

$$Y_n = -G_n + J_n(\log 2 - \gamma + \tfrac{1}{2}i\pi)$$
$$= -\frac{\pi}{2\sin n\pi}\{J_{-n} - \cos n\pi\, J_n\} + J_n(\log 2 - \gamma). \tag{70}$$

By means of this equation Y_n is defined even when n is not an integer.

From (67) it is clear that

$$G_{-n} = e^{in\pi}G_n \quad . \quad . \quad . \tag{71}$$

From this equation, along with (69), the value of G_n, and consequently of Y_n, can be found when n is a negative integer.

Recurrence Formulæ. From (67) it follows that

$$xG'_n = \frac{\pi}{2\sin n\pi}(xJ'_{-n} - e^{-in\pi}xJ'_n),$$

and, from (32) and (31), that

$$xJ'_{-n} = nJ_{-n} + xJ_{-n-1}, \quad xJ'_n = nJ_n - xJ_{n+1};$$

hence

$$\begin{aligned} xG'_n &= n\frac{\pi}{2\sin n\pi}\{J_{-n} - e^{-in\pi}J_n\} \\ &\qquad - x\frac{\pi}{2\sin(n+1)\pi}\{J_{-n-1} - e^{-i(n+1)\pi}J_{n+1}\} \\ &= nG_n - xG_{n+1}. \qquad (72) \end{aligned}$$

Again, from (31) and (32),

$$xJ'_{-n} = -nJ_{-n} - xJ_{-n+1}, \quad xJ'_n = -nJ_n + xJ_{n-1},$$

so that

$$\begin{aligned} xG'_n &= -n\frac{\pi}{2\sin n\pi}\{J_{-n} - e^{-in\pi}J_n\} \\ &\qquad + x\frac{\pi}{2\sin(n-1)\pi}\{J_{-n+1} - e^{-i(n-1)\pi}J_{n-1}\} \\ &= -nG_n + xG_{n-1}. \qquad (73) \end{aligned}$$

Thus G_n satisfies the recurrence formulæ (31) and (32), and consequently also (33), (34), (35), (36), (37), and (38).

Note.—Since any other Bessel Function, such as Y_n, is of the form

$$AJ_n + BG_n,$$

it will also satisfy these recurrence formulæ.

§ 9. **Relations between the Bessel Functions.** If $P(x)$ and $Q(x)$ are any Bessel Functions of order n, they satisfy a relation of the form

$$PQ' - P'Q = C/x, \qquad (74)$$

where C is a constant.

For, since P and Q satisfy Bessel's Equation,

$$\frac{d}{dx}(xP') = (n^2 - x^2)\frac{P}{x},$$

and

$$\frac{d}{dx}(xQ') = (n^2 - x^2)\frac{Q}{x}.$$

Hence $$P\frac{d}{dx}(xQ') - Q\frac{d}{dx}(xP') = 0,$$

or $$\frac{d}{dx}\{x(PQ' - P'Q)\} = 0;$$

so that $$x(PQ' - P'Q) = C.$$

In particular, if $P = J_n$, $Q = J_{-n}$,

$$\lim_{x \to 0} x(J_n J'_{-n} - J'_n J_{-n}) = -\frac{1}{\Gamma(n)\Gamma(1-n)} + \frac{1}{\Gamma(-n)\Gamma(1+n)}$$
$$= -\frac{2}{\pi}\sin n\pi,$$

and therefore

$$J_n J'_{-n} - J'_n J_{-n} = -\frac{2\sin n\pi}{\pi x}. \quad . \quad . \quad (75)$$

It is left to the reader to deduce that

$$G_n J'_n - G'_n J_n = J_n Y'_n - J'_n Y_n = \frac{1}{x}, \quad . \quad . \quad (76)$$

$$J_n J_{-n+1} + J_{-n} J_{n-1} = -J_n J_{-n-1} - J_{-n} J_{n+1} = \frac{2\sin n\pi}{\pi x}, \quad (77)$$

$$G_{n+1} J_n - G_n J_{n+1} = J_{n+1} Y_n - Y_{n+1} J_n = \frac{1}{x}. \quad (78)$$

§ 10. **Addition Theorems for Y_n and G_n.** An addition formula similar to (50) will now be established for the functions Y_n and G_n, when n is an integer.

Since, when n is zero or a positive integer, $Y_n(x)$ is a real function of x which is continuous except at $x = 0$,

$$Y_n(u + v) = \sum_{r=0}^{\infty} \frac{v^r}{r!}\frac{d^r Y_n(u)}{du^r},$$

provided that $|v| < |u|$. Employing formula (38) we can write this

$$Y_n(u + v) = \sum_{r=0}^{\infty}\left(\frac{v}{2}\right)^r \sum_{s=0}^{r} \frac{(-1)^s}{\Gamma(s+1)\Gamma(r-s+1)} Y_{n-r+2s}(u),$$

and, since $1/\Gamma(x)$ vanishes when x is zero or a negative integer, this can be written

$$Y_n(u + v) = \sum_{r=0}^{\infty}\left(\frac{v}{2}\right)^r \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(r-s+1)} Y_{n-r+2s}(u),$$

or, taking the terms in which r is even or odd by themselves,

$$Y_n(u+v) = \sum_{r=0}^{\infty} \left(\frac{v}{2}\right)^{2r} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(2r-s+1)} Y_{n-2r+2s}(u)$$

$$+ \sum_{r=0}^{\infty} \left(\frac{v}{2}\right)^{2r+1} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(2r-s+2)} Y_{n-2r+2s-1}(u).$$

In these summations put $s = p + r$, and change the order of summation; thus

$$Y_n(u+v)$$

$$= \sum_{p=-\infty}^{\infty} (-1)^p Y_{n+2p}(u) \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+1)} \left(\frac{v}{2}\right)^{2r}$$

$$+ \sum_{p=-\infty}^{\infty} (-1)^p Y_{n+2p-1}(u) \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+2)} \left(\frac{v}{2}\right)^{2r+1},$$

and, since $1/\Gamma(x)$ vanishes when x is zero or a negative integer,

$$Y_n(u+v)$$

$$= \sum_{p=0}^{\infty} (-1)^p Y_{n+2p}(u) \sum_{r=p}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+1)} \left(\frac{v}{2}\right)^{2r}$$

$$+ \sum_{p=-\infty}^{-1} (-1)^p Y_{n+2p}(u) \sum_{r=-p}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+1)} \left(\frac{v}{2}\right)^{2r}$$

$$+ \sum_{p=1}^{\infty} (-1)^p Y_{n+2p-1}(u) \sum_{r=p-1}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+2)} \left(\frac{v}{2}\right)^{2r+1}$$

$$+ \sum_{p=-\infty}^{0} (-1)^p Y_{n+2p-1}(u) \sum_{r=-p}^{\infty} \frac{(-1)^r}{\Gamma(p+r+1)\Gamma(-p+r+2)} \left(\frac{v}{2}\right)^{2r+1}$$

$$= \sum_{p=0}^{\infty} Y_{n+2p}(u) \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(2p+s+1)\Gamma(s+1)} \left(\frac{v}{2}\right)^{2p+2s}$$

$$+ \sum_{p=1}^{\infty} Y_{n-2p}(u) \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(2p+s+1)} \left(\frac{v}{2}\right)^{2p+2s}$$

$$-\sum_{p=1}^{\infty} Y_{n+2p-1}(u) \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(2p+s)\Gamma(s+1)}\left(\frac{v}{2}\right)^{2p+2s-1}$$

$$+\sum_{p=0}^{\infty} Y_{n-2p-1}(u) \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(2p+s+2)}\left(\frac{v}{2}\right)^{2p+2s+1}$$

$$=\sum_{p=0}^{\infty} Y_{n+2p}(u)J_{2p}(v) + \sum_{p=1}^{\infty} Y_{n-2p}(u)J_{2p}(v)$$

$$-\sum_{p=1}^{\infty} Y_{n+2p-1}(u)J_{2p-1}(v) + \sum_{p=0}^{\infty} Y_{n-2p-1}(u)J_{2p+1}(v)$$

$$=\sum_{p=-\infty}^{\infty} Y_{n-p}(u)J_p(v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

Since G_n is of the form $AJ_n + BY_n$, it follows from (50) and (79) that

$$G_n(u+v) = \sum_{p=-\infty}^{\infty} G_{n-p}(u)J_p(v), \quad . \quad . \qquad (80)$$

where n is zero or a positive integer, and $|v| < |u|$.

From (71) and (21) it follows that (80) also holds when n is a negative integer; consequently (79) will then be valid also.

Examples.

1. Verify that $x^{-\frac{1}{2}n}J_n(2\sqrt{x})$ and $x^{-\frac{1}{2}n}Y_n(2\sqrt{x})$ are solutions of

$$xy'' + (n+1)y' + y = 0.$$

2. Show that every solution of

$$y'' + xy = 0$$

can be put in the form

$$Ax^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}x^{\frac{3}{2}}) + Bx^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}x^{\frac{3}{2}}).$$

3. Show that the general solution of

$$x^2y'' - 2xy' + 4(x^4 - 1)y$$

is

$$Ax^{\frac{3}{2}}J_{\frac{5}{4}}(x^2) + Bx^{\frac{1}{2}}J_{-\frac{5}{4}}(x^2).$$

4. Prove that the general solution of

$$x^2y'' + (2\alpha - 2\beta n + 1)xy' + \{\beta^2\gamma^2x^{2\beta} + \alpha(\alpha - 2\beta n)\}y = 0$$

is $$x^{\beta n - \alpha}\{AJ_n(\gamma x^\beta) + BY_n(\gamma x^\beta)\}.$$

5. Show that

(i) $J_2 - J_0 = 2J_0''$; (ii) $J_2 = J_0'' - x^{-1}J_0'$;

(iii) $J_3 + 3J_0' + 4J_0''' = 0$.

6. Prove that

(i) $\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$,

(ii) $\frac{d}{dx}(xJ_nJ_{n+1}) = x(J_n^2 - J_{n+1}^2)$.

Deduce that

(iii) $1 = J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots$,

(iv) $x = 2J_0J_1 + 6J_1J_2 + \dots + 2(2n + 1)J_nJ_{n+1} + \dots$.

7. If $n > -1$, show that

(i) $\int_0^x x^{n+1}J_n(x)dx = x^{n+1}J_{n+1}(x)$;

(ii) $\int_0^x x^{-n}J_{n+1}(x)dx = \frac{1}{2^n\Gamma(n + 1)} - x^{-n}J_n(x)$.

8. Show that $$AJ_n(x)\int^x \frac{dx}{xJ_n^2(x)} + BJ_n(x)$$

is the complete solution of Bessel's Equation.

9. Show that

(i) $\sqrt{(\frac{1}{2}\pi x)}J_{\frac{3}{2}}(x) = \frac{\sin x}{x} - \cos x$;

(ii) $\sqrt{(\frac{1}{2}\pi x)}J_{-\frac{3}{2}}(x) = -\sin x - \frac{\cos x}{x}$.

10. Prove that

$$J_{n+1}(x) = x\int_0^1 J_n(xy)y^{n+1}dy.$$

11. Show that

(i) $\int_0^y \frac{x\sin ax}{\sqrt{(y^2 - x^2)}}dx = \frac{1}{2}\pi y J_1(ay)$,

(ii) $\int_0^{\frac{\pi}{2}} J_1(y\sin\theta)d\theta = \frac{1 - \cos y}{y}$.

12. Prove that, if $n > m > -1$,

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^{n-m}}{\Gamma(n - m)}\int_0^1 u^{\frac{1}{2}m}(1 - u)^{n-m-1}J_m(x\sqrt{u})du.$$

13. Show that

$$\int_0^1 \frac{uJ_0(xu)du}{\sqrt{(1 - u^2)}} = \frac{\sin x}{x}.$$

14. Prove that

(i) $x^2J_n'' = (n^2 - n - x^2)J_n + xJ_{n+1}$,

(ii) $x^2G_n'' = (n^2 - n - x^2)G_n + xG_{n+1}$.

15. If $P_n(x)$ and $Q_m(x)$ are Bessel Functions of orders n and m respectively, show that

$$(n^2 - m^2)\int_a^b P_n(x)Q_m(x)\frac{dx}{x} = \Big[x\{Q_m(x)P_n'(x) - P_n(x)Q_m'(x)\}\Big]_a^b.$$

[In Bessel's Equation (12) put $y = x^{-\frac{1}{2}}z$, and it becomes

$$z'' + \left(1 - \frac{4n^2 - 1}{4x^2}\right)z = 0.$$

Thus, if $P_n(x) = x^{-\frac{1}{2}}R_n(x)$, and $Q_m(x) = x^{-\frac{1}{2}}S_m(x)$,

$$R_n'' + \left(1 - \frac{4n^2 - 1}{4x^2}\right)R_n = 0, \text{ and } S_m'' + \left(1 - \frac{4m^2 - 1}{4x^2}\right)S_m = 0.$$

Now multiply these equations by S_m and R_n, substract, and get

$$\frac{d}{dx}(S_mR_n' - R_nS_m') = -\frac{n^2 - m^2}{x^2}R_nS_m,$$

from which the required result follows.]

CHAPTER XV

ASYMPTOTIC EXPANSIONS AND FOURIER-BESSEL EXPANSIONS

§ 1. Additional Definite Integral Expressions for the Bessel Functions. In (XIV., 59) let $\lambda = -2v \pm i$; then the functions

$$y = x^n e^{\pm ix}\int e^{-2vx}(v \pm iv^2)^{n-\frac{1}{2}}dv \quad . \quad . \quad (1)$$

will be solutions of Bessel's Equation, provided that $e^{-2vx \pm ix}(v \pm iv^2)^{n+\frac{1}{2}}$ has equal values at the limits of integration. If $n > -\frac{1}{2}$, we can take the limits to be 0 and $+\infty$, and so obtain the solution

$$y = x^n\left\{Ae^{ix}\int_0^\infty e^{-2vx}(v+iv^2)^{n-\frac{1}{2}}dv + Be^{-ix}\int_0^\infty e^{-2vx}(v-iv^2)^{n-\frac{1}{2}}dv\right\}, \quad (2)$$

where $x > 0$.

In (2) put $v = \tan\theta$, and it becomes

$$y = x^n\int_0^{\frac{\pi}{2}}\frac{(\sin\theta)^{n-\frac{1}{2}}}{(\cos\theta)^{2n+1}}\{Ae^{i(n-\frac{1}{2})\theta + ix} + Be^{-i(n-\frac{1}{2})\theta - ix}\}e^{-2x\tan\theta}d\theta;$$

or, if θ be replaced by $\frac{1}{2}\pi - \theta$,

$$y = x^n\int_0^{\frac{\pi}{2}}\frac{(\cos\theta)^{n-\frac{1}{2}}}{(\sin\theta)^{2n+1}}\left[\begin{array}{l}Ae^{\frac{1}{2}(n-\frac{1}{2})\pi i + i\{x-(n-\frac{1}{2})\theta\}}\\ \quad + Be^{-\frac{1}{2}(n-\frac{1}{2})\pi i - i\{x-(n-\frac{1}{2})\theta\}}\end{array}\right]e^{-2x\cot\theta}d\theta.$$

If we here put $A = -ie^{-\frac{1}{2}(n-\frac{1}{2})\pi i}$, $B = ie^{\frac{1}{2}(n-\frac{1}{2})\pi i}$, (3)
we obtain the Bessel Function

$$Z_n = 2x^n\int_0^{\frac{\pi}{2}}\frac{(\cos\theta)^{n-\frac{1}{2}}}{(\sin\theta)^{2n+1}}\sin\{x-(n-\tfrac{1}{2})\theta\}e^{-2x\cot\theta}d\theta, \quad (4)$$

where $n > -\frac{1}{2}$, $x > 0$.

It should be noted that, since Z_n is a Bessel Function, it satisfies the recurrence formulæ of Ch. XIV., § 3, so long as $n > -\frac{1}{2}$ [Cf. Ch. XIV., § 8, *Note*].

It will now be shown that the function Z_n is identical with the function J_n. When $-\frac{1}{2} < n < \frac{1}{2}$, the integral in (4) is still convergent when x tends to zero: hence

$$\begin{aligned}\operatorname*{Lim}_{x\to 0}\frac{Z_n}{x^n} &= 2\int_0^{\frac{\pi}{2}}\frac{(\cos\theta)^{n-\frac{1}{2}}\sin\left(\frac{1-2n}{2}\theta\right)}{(\sin\theta)^{2n+1}}d\theta\\ &= 2\int_0^{\frac{\pi}{2}}(\cos\theta)^{-m}(\sin\theta)^{2m-2}\sin m\theta\ d\theta,\end{aligned}$$

where $m = \frac{1}{2} - n$ lies between 0 and 1.

Now, from (IV., 25),

$$\sin m\theta = m\sin\theta\,\mathrm{F}\left(\frac{1+m}{2}, \frac{1-m}{2}, \frac{3}{2}, \sin^2\theta\right),$$

and therefore, if $u = \sin^2\theta$,

$$\operatorname*{Lim}_{x\to 0}\frac{Z_n}{x^n} = m\int_0^1 u^{m-1}(1-u)^{-\frac{1+m}{2}}\mathrm{F}\left(\frac{1+m}{2}, \frac{1-m}{2}, \frac{3}{2}, u\right)du.$$

If this expression be integrated term by term, it gives

$$\begin{aligned}\operatorname*{Lim}_{x\to 0}\frac{Z_n}{x^n} &= m\mathrm{B}\left(m, \frac{1-m}{2}\right)\\ &\quad\times\left\{1 + \frac{m(1-m)}{1\cdot 3} + \frac{m(m+1)(1-m)(3-m)}{2!\,3\cdot 5} + \dots\right\}\\ &= m\frac{\Gamma(m)\Gamma\left(\frac{1-m}{2}\right)}{\Gamma\left(\frac{1+m}{2}\right)}\mathrm{F}\left(\frac{1-m}{2}, m, \frac{3}{2}, 1\right)\\ &= \frac{\Gamma(m+1)\Gamma\left(\frac{1-m}{2}\right)}{\Gamma\left(\frac{1+m}{2}\right)}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1-\frac{m}{2}\right)}{\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(\frac{3}{2}-m\right)},\end{aligned}$$

by (IV., 56). But, by (IV., 41),

$$\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(1-\frac{m}{2}\right) = \sqrt{\pi}\Gamma(1-m)2^m,$$

$$\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(1+\frac{m}{2}\right) = \sqrt{\pi}\Gamma(1+m)2^{-m};$$

therefore

$$\operatorname*{Lim}_{x\to 0}\frac{Z_n}{x^n} = 2^{2m-1}\sqrt{\pi}\frac{\Gamma(1-m)}{\Gamma(\frac{3}{2}-m)} = \sqrt{\pi}\frac{\Gamma(n+\frac{1}{2})}{2^{2n}\Gamma(n+1)}. \qquad (5)$$

Now, since Z_n is a Bessel Function

$$Z_n = CJ_n + DJ_{-n},$$

and, consequently,

$$\operatorname*{Lim}_{x \to 0} \frac{Z_n}{x^n} = C\frac{1}{2^n\Gamma(n+1)} + D \operatorname*{Lim}_{x \to 0} \frac{J_{-n}}{x^n}. \quad . \quad (6)$$

But, if $0 < n < \frac{1}{2}$, the second limit is infinite, and therefore D must be zero. Hence, from (5),

$$C = \sqrt{\pi} \,.\, \Gamma(n + \tfrac{1}{2})2^{-n},$$

so that $Z_n = \sqrt{\pi} \,.\, \Gamma(n + \frac{1}{2})2^{-n}J_n$, and therefore, from (4), if $0 < n < \frac{1}{2}$,

$$J_n(x) = \frac{2^{n+1}x^{-n}}{\sqrt{\pi} \,.\, \Gamma(n + \frac{1}{2})}\int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{n-\frac{1}{2}}}{(\sin\theta)^{2n+1}} \sin\left(x - \frac{2n-1}{2}\theta\right)e^{-2x\cot\theta}d\theta. \tag{7}$$

If $-\frac{1}{2} < n < 0$, the second limit on the right of (6) has the value zero, so that, while C has the same value as before, it does not necessarily follow that D is zero. To show that D is zero in this case also, integrate in (4) by parts, and get

$$\begin{aligned} Z_n = x^{n-1}&\left[\frac{(\cos\theta)^{n-\frac{1}{2}}}{(\sin\theta)^{2n-1}} \sin\left(x - \frac{2n-1}{2}\theta\right)e^{-2x\cot\theta}\right]_0^{\frac{\pi}{2}} \\ &+ (n - \tfrac{1}{2})x^{n-1}\int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{n-\frac{3}{2}}}{(\sin\theta)^{2n-1}} \cos\left(x - \frac{2n+1}{2}\theta\right)e^{-2x\cot\theta}d\theta \\ &+ (2n-1)x^{n-1}\int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{n+\frac{1}{2}}}{(\sin\theta)^{2n}} \sin\left(x - \frac{2n-1}{2}\theta\right)e^{-2x\cot\theta}d\theta. \end{aligned}$$

In this equation the first term on the right vanishes if $n > \frac{1}{2}$, and the two integrals are still convergent when x vanishes, if $n < 1$. Thus the value of $\operatorname*{Lim}_{x \to 0}(x^{1-n}Z_n)$ is finite if $\frac{1}{2} < n < 1$. But $\operatorname*{Lim}_{x \to 0}(x^{1-n}J_{-n})$ is infinite when $\frac{1}{2} < n < 1$; hence, for these values of n, Z_n is a constant multiple of J_n; say $Z_n = KJ_n$.

But, from (XIV., 32)

$$J_{n-1} = \frac{n}{x}J_n + J_n',$$

and, since Z_n possesses a derivative when $x > 0$,

$$Z_{n-1} = \frac{n}{x}Z_n + Z_n' = K\left\{\frac{n}{x}J_n + J_n'\right\} = KJ_{n-1};$$

so that, if $-\frac{1}{2} < n < 0$, Z_n is a constant multiple of J_n, and therefore (7) holds in this case also. Since both sides of (7) are continuous functions of n, this formula also holds when $n = 0$, and when $n = \frac{1}{2}$. Also, since Z_n and J_n both satisfy the formula (XIV., 31)

$$J_{n+1} = \frac{n}{x}J_n - J_n',$$

it follows that (7) holds for $\frac{1}{2} < n \leqq \frac{3}{2}$, then for $\frac{3}{2} < n \leqq \frac{5}{2}$, and so for all values of $n > -\frac{1}{2}$, provided that $x > 0$.

From (7), (4), (3), and (2) it follows that, if $x > 0$ and $n > -\frac{1}{2}$,

$$J_n(x) = \frac{(2x)^n}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left\{\begin{array}{l} e^{ix-(n+\frac{1}{2})\frac{1}{2}\pi i}\int_0^\infty e^{-2vx}(v+iv^2)^{n-\frac{1}{2}}dv \\ +e^{-ix+(n+\frac{1}{2})\frac{1}{2}\pi}\int_0^\infty e^{-2vx}(v-iv^2)^{n-\frac{1}{2}}dv \end{array}\right\}.$$

Here put $v = u/(2x)$, and so obtain the formula

$$J_n(x) = \frac{1}{\sqrt{(2\pi x)}\Gamma(n+\frac{1}{2})} \times \left\{\begin{array}{l} e^{ix-(n+\frac{1}{2})\frac{1}{2}\pi i}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}du \\ +e^{-ix+(n+\frac{1}{2})\frac{1}{2}\pi i}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1-\frac{iu}{2x}\right)^{n-\frac{1}{2}}du \end{array}\right\}. \qquad (8)$$

Again, from (1), by writing $v = u/(2x)$, we see that

$$U_n(x) = e^{-(n-\frac{1}{2})\frac{1}{2}\pi i}\frac{1}{\Gamma(n+\frac{1}{2})}\sqrt{\left(\frac{\pi}{2x}\right)}e^{ix}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}du, \qquad (9)$$

where $n > -\frac{1}{2}$ and $x > 0$, is a Bessel Function, and it will now be shown that this is the function $G_n(x)$.

Let $$U_n = AJ_n + BJ_{-n}, \quad . \quad . \quad . \quad (10)$$

where $-\frac{1}{2} < n < \frac{1}{2}$; then, if, in (8) and (9), we put $x = 2p\pi + y$, where p is an integer and $0 < y \leqq 2\pi$, we see that

$$\lim_{p\to\infty}\{\sqrt{(2\pi x)}J_n(x)\} = 2\cos\{y - (n+\tfrac{1}{2})\tfrac{1}{2}\pi\}$$

and $$\lim_{p\to\infty}\{\sqrt{(2\pi x)}U_n(x)\} = \pi e^{i\{y-(n-\frac{1}{2})\frac{1}{2}\pi\}}.$$

Thus, if we multiply (10) by $\surd(2\pi x)$, and make p tend to infinity, we find that, for $-\frac{1}{2} < n < \frac{1}{2}$,

$$\begin{aligned}\pi e^{i\{y-(n-\frac{1}{2})\frac{1}{2}\pi\}} &= \mathrm{A}2\cos\{y-(n+\tfrac{1}{2})\tfrac{1}{2}\pi\}+\mathrm{B}2\cos\{y+(n-\tfrac{1}{2})\tfrac{1}{2}\pi\}\\ &= \mathrm{A}2\sin\{y-(n-\tfrac{1}{2})\tfrac{1}{2}\pi\}+\mathrm{B}2\cos\{y+(n-\tfrac{1}{2})\tfrac{1}{2}\pi\}.\end{aligned}$$

On equating the coefficients of $\cos y$ and $\sin y$ in this equation, we obtain the equations

$$\pi e^{-(n-\frac{1}{2})\frac{1}{2}\pi i} = -2\mathrm{A}\sin\left(\frac{2n-1}{4}\pi\right)+2\mathrm{B}\cos\left(\frac{2n-1}{4}\pi\right),$$

$$i\pi e^{-(n-\frac{1}{2})\frac{1}{2}\pi i} = 2\mathrm{A}\cos\left(\frac{2n-1}{4}\pi\right)-2\mathrm{B}\sin\left(\frac{2n-1}{4}\pi\right);$$

and, on solving for A and B, we find that

$$\mathrm{A} = -\frac{\pi}{2\sin n\pi}e^{-in\pi}, \quad \mathrm{B} = \frac{\pi}{2\sin n\pi},$$

so that

$$\mathrm{U}_n = \frac{\pi}{2\sin n\pi}(\mathrm{J}_{-n} - e^{-in\pi}\mathrm{J}_n) = \mathrm{G}_n,$$

where $-\frac{1}{2} < n < \frac{1}{2}$; in the same way as for J_n, it can be shown that this identity is valid for $n > -\frac{1}{2}$.

Hence, from (9),

$$\mathrm{G}_n(x) = e^{-(n-\frac{1}{2})\frac{1}{2}\pi i}\frac{1}{\Gamma(n+\frac{1}{2})}\sqrt{\left(\frac{\pi}{2x}\right)}e^{ix}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}du, \qquad (11)$$

where $n > -\frac{1}{2}$ and $x > 0$.

§ 2. **The Modified Bessel Functions I_n and K_n.** If in Bessel's Equation (XIV., 12) we put $x = it$, it becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t^2+n^2)y = 0, \quad . \quad . \quad (12)$$

of which $\mathrm{J}_n(it)$ and $\mathrm{G}_n(it)$ are solutions. Instead of these, however, it is usual to take as solutions the functions

$$\mathrm{I}_n(t) \equiv e^{-\frac{1}{2}n\pi i}\mathrm{J}_n(it), \quad . \quad . \quad . \quad (13)$$

and

$$\mathrm{K}_n(t) \equiv e^{\frac{1}{2}n\pi i}\mathrm{G}_n(it). \quad . \quad . \quad . \quad (14)$$

From (XIV., 19) and (XIV., 67) it follows that

$$\mathrm{I}_n(t) = \sum_{r=0}^{\infty}\frac{1}{\Gamma(r+1)\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r} \quad . \quad (15)$$

and

$$K_n(t) = \frac{\pi}{2 \sin n\pi}\{I_{-n}(t) - I_n(t)\}; \quad . \quad . \quad (16)$$

so that

$$K_{-n}(t) = K_n(t). \quad . \quad . \quad . \quad (17)$$

By means of (13) and (14) numerous formulæ for I_n and K_n may be deduced from the corresponding formulæ for J_n and G_n. The verification of these is left to the reader.

From (XIV., 21), if n is an integer,

$$I_{-n}(t) = I_n(t). \quad . \quad . \quad . \quad (18)$$

From (XIV., 68), if n is a positive integer,

$$K_n(t) = (-1)^{n+1} I_n(t) \log\left(\frac{t}{2}\right) + \frac{1}{2}\sum_{r=0}^{n-1} \frac{(-1)^r (n-r-1)!}{r!}\left(\frac{t}{2}\right)^{-n+2r}$$

$$+ (-1)^n \frac{1}{2} \sum_{r=0}^{\infty} \frac{1}{r!\,(n+r)!}\left(\frac{t}{2}\right)^{n+2r} \{\phi(r) + \phi(n+2r) - 2\gamma\}. \quad (19)$$

From the formulæ (31) to (38) of the previous chapter we deduce that

$$tI'_n = nI_n + tI_{n+1}, \quad . \quad . \quad . \quad (20)$$

$$tI'_n = -nI_n + tI_{n-1}, \quad . \quad . \quad (21)$$

$$2I'_n = I_{n-1} + I_{n+1}, \quad . \quad . \quad (22)$$

$$I'_0 = I_1, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{2n}{t}I_n = I_{n-1} - I_{n+1}, \quad . \quad . \quad . \quad (24)$$

$$\frac{d}{dt}(t^n I_n) = t^n I_{n-1}, \quad . \quad . \quad . \quad . \quad (25)$$

$$\frac{d}{dt}(t^{-n} I_n) = t^{-n} I_{n+1}. \quad . \quad . \quad . \quad . \quad (26)$$

$$2^r \frac{d^r}{dt^r} I_n = I_{n-r} + rI_{n-r+2} + \frac{r(r-1)}{2!} I_{n-r+4} + \dots + I_{n+r} \quad (27)$$

The corresponding formulæ for G_n (XIV., § 8) lead to

$$tK'_n = nK_n - tK_{n+1}, \quad . \quad . \quad . \quad (28)$$

$$tK'_n = -nK_n - tK_{n-1}, \quad . \quad . \quad (29)$$

$$2K'_n = -K_{n-1} - K_{n+1}, \quad . \quad . \quad (30)$$

$$K'_0 = -K_1, \quad . \quad . \quad . \quad . \quad (31)$$

$$\frac{2n}{t}\mathrm{K}_n = \mathrm{K}_{n+1} - \mathrm{K}_{n-1}, \quad . \quad . \quad . \quad (32)$$

$$\frac{d}{dt}(t^n\mathrm{K}_n) = -t^n\mathrm{K}_{n-1}, \quad . \quad . \quad . \quad (33)$$

$$\frac{d}{dt}(t^{-n}\mathrm{K}_n) = -t^{-n}\mathrm{K}_{n+1}, \quad . \quad . \quad . \quad (34)$$

$$(-2)^r\frac{d^r}{dt^r}\mathrm{K}_n = \mathrm{K}_{n-r} + r\mathrm{K}_{n-r+2} + \frac{r(r-1)}{2\,!}\mathrm{K}_{n-r+4} + \ . \ . \ . \ + \mathrm{K}_{n+r} \quad (35)$$

From (75), (76), (77), and (78) of Chapter XIV. we deduce that

$$\mathrm{I}_n\mathrm{I}'_{-n} - \mathrm{I}'_n\mathrm{I}_{-n} = -\frac{2\sin n\pi}{\pi t}, \quad . \quad . \quad (36)$$

$$\mathrm{I}_n\mathrm{I}_{-n+1} - \mathrm{I}_{-n}\mathrm{I}_{n-1} = \mathrm{I}_n\mathrm{I}_{-n-1} - \mathrm{I}_{-n}\mathrm{I}_{n+1} = -\frac{2\sin n\pi}{\pi t}, \quad (37)$$

$$\mathrm{K}_n\mathrm{I}'_n - \mathrm{K}'_n\mathrm{I}_n = \mathrm{K}_{n+1}\mathrm{I}_n + \mathrm{K}_n\mathrm{I}_{n+1} = \frac{1}{t}. \quad . \quad (38)$$

From (XIV., 50) it follows that, when n is an integer,

$$\mathrm{I}_n(u+v) = \sum_{r=-\infty}^{\infty}\mathrm{I}_r(u)\mathrm{I}_{n-r}(v), \quad . \quad . \quad (39)$$

and, from (XIV., 53), that

$$\mathrm{I}_0\{\sqrt{(b^2+c^2-2bc\cos\alpha)}\} = \mathrm{I}_0(b)\mathrm{I}_0(c)$$
$$+ 2\sum_{n=1}^{\infty}(-1)^n\mathrm{I}_n(b)\mathrm{I}_n(c)\cos n\alpha\,; \quad (40)$$

while (XIV., 80) gives, when $|v| < |u|$ and $n = 0, 1, 2, \ldots$

$$\mathrm{K}_n(u+v) = \sum_{r=-\infty}^{\infty}(-1)^r\mathrm{K}_{n-r}(u)\mathrm{I}_r(v). \quad . \quad (41)$$

Again, from formulæ (62) to (66) of Chapter XIV.,

$$\mathrm{I}_n(t) = \frac{1}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{t}{2}\right)^n\int_{-1}^{1}e^{\pm\lambda t}(1-\lambda^2)^{n-\frac{1}{2}}d\lambda, \quad . \quad (42)$$

$$= \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{t}{2}\right)^n\int_0^1\cosh(\lambda t)(1-\lambda^2)^{n-\frac{1}{2}}d\lambda, \quad (43)$$

$$= \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{t}{2}\right)^n\int_0^{\frac{\pi}{2}}\cosh(t\sin\phi)(\cos\phi)^{2n}d\phi \quad (44)$$

$$= \frac{2}{\sqrt{\pi}\,.\,\Gamma(n+\frac{1}{2})}\left(\frac{t}{2}\right)^n\int_0^{\frac{\pi}{2}}\cosh(t\cos\phi)(\sin\phi)^{2n}d\phi. \quad (45)$$

Finally, from (8) and (11), we have, if $x > 0$ and $n > -\frac{1}{2}$,

$$\mathrm{I}_n(x) = \frac{1}{\sqrt{(2\pi x)}\Gamma(n+\frac{1}{2})} \times \left\{ \begin{array}{l} e^{-x-(n+\frac{1}{2})\pi i}\displaystyle\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{u}{2x}\right)^{n-\frac{1}{2}}du \\ \qquad + e^x\displaystyle\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1-\frac{u}{2x}\right)^{n-\frac{1}{2}}du \end{array} \right\}, \tag{46}$$

and

$$\mathrm{K}_n(x) = e^{-x}\sqrt{\left(\frac{\pi}{2x}\right)}\frac{1}{\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{u}{2x}\right)^{n-\frac{1}{2}}du. \tag{47}$$

The importance of the function $\mathrm{K}_n(x)$ in applications to Physics lies in the fact, evident from this formula, that it tends to zero when x tends to infinity.

§ 3. **The Asymptotic Expansions.** In Chapter IV., § 10, it was shown that the binomial expansion can be put in the form

$$(1+z)^m = \sum_{r=0}^{s-1}\frac{\Gamma(m+1)}{r!\,\Gamma(m-r+1)}z^r + \mathrm{R}'_s, \quad . \tag{48}$$

where

$$\mathrm{R}'_s = \frac{\Gamma(m+1)}{s!\,\Gamma(m-s+1)}z^s\int_0^1 s(1-t)^{s-1}(1+zt)^{m-s}dt, \tag{49}$$

provided that $(1+zt)$ does not vanish for $0 \leqq t \leqq 1$.

Now apply this result to the binomial expression $\left(1+\frac{u}{2x}\right)^{n-\frac{1}{2}}$ in (47); then, integrating term by term, we have

$$\mathrm{K}_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)}e^{-x}\left[\sum_{r=0}^{s-1}\frac{\Gamma(n+r+\frac{1}{2})}{r!\,\Gamma(n-r+\frac{1}{2})}\frac{1}{(2x)^r} + \mathrm{R}_s\right], \tag{50}$$

where $x > 0$ and

$$\mathrm{R}_s = \frac{1}{s!\,\Gamma(n-s+\frac{1}{2})(2x)^s} \times \int_0^\infty e^{-u}u^{n+s-\frac{1}{2}}du\int_0^1 s(1-t)^{s-1}\left(1+\frac{ut}{2x}\right)^{n-s-\frac{1}{2}}dt. \tag{51}$$

Hence, if x is real and positive, and s is chosen so large that $n-s-\frac{1}{2}$ is negative, since $1+\frac{ut}{2x} \geqq 1$,

$$\left(1+\frac{ut}{2x}\right)^{n-s-\frac{1}{2}} \leqq 1,$$

and therefore

$$\begin{aligned}|\mathrm{R}_s| &\leq \frac{1}{s!\,\Gamma(n-s+\frac{1}{2})(2x)^s}\int_0^\infty e^{-u}u^{n+s-\frac{1}{2}}du\int_0^1 s(1-t)^{s-1}dt\\ &= \frac{\Gamma(n+s+\frac{1}{2})}{s!\,\Gamma(n-s+\frac{1}{2})}\,\frac{1}{(2x)^s}, \qquad \text{. . . . . .} \qquad (52)\end{aligned}$$

which is the $(s+1)$th term in the series in (50). But, by making x large enough, this remainder can be made arbitrarily small. An expansion of this kind, consisting of a finite number of terms and a remainder which can be made arbitrarily small by increasing the variable is called an Asymptotic Expansion.

The expansion (50) can be written

$$\mathrm{K}_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)}e^{-x}\left\{1 + \frac{4n^2-1^2}{1!\,8x} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8x)^2} + \dots\right\}. \qquad (53)$$

Since $\mathrm{K}_{-n} = \mathrm{K}_n$, this expansion still holds when n is negative.

The expansion also holds for certain complex values of x, and, indeed, it can be shown* by the method of contour integration that it holds if $x \neq 0$, $-\frac{3}{2}\pi < \operatorname{amp} x < \frac{3}{2}\pi$.

From the formula

$$i\mathrm{G}_n(x) = e^{-\frac{1}{2}n\pi i}\mathrm{K}_n(e^{-\frac{1}{2}\pi i}x), \qquad \text{. . .} \qquad (54)$$

or directly from (11), it can be deduced that the asymptotic expansion of $\mathrm{G}_n(x)$ is

$$\begin{aligned}\mathrm{G}_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)}e^{-\frac{1}{2}n\pi i + i(x+\frac{1}{4}\pi)}\\ \times\left[\begin{array}{l}\left\{1 - \dfrac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8x)^2} + \dfrac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!\,(8x)^4} - \dots\right\}\\ + i\left\{\dfrac{4n^2-1^2}{1!\,8x} - \dfrac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8x)^3} + \dots\right\}\end{array}\right], \qquad (55)\end{aligned}$$

where $x \neq 0$, $-\pi < \operatorname{amp} x < 2\pi$.

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* See *Bessel Functions* by Gray, Mathews, and MacRobert, pp. 55, 56, xiii, xiv.

Again, from the definition of $G_n(x)$, (XIV., 67),

$$\pi i J_n(x) = G_n(x) - e^{in\pi} G_n(xe^{i\pi}), \qquad (56)$$

and therefore, if $x \neq 0$, $-\pi < \operatorname{amp} x < \pi$, the asymptotic expansion of $J_n(x)$ is

$$\begin{aligned} J_n(x) = {} & \sqrt{\left(\frac{2}{\pi x}\right)} \cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\left\{1 - \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\,(8x)^2} + \ldots\right\} \\ & - \sqrt{\left(\frac{2}{\pi x}\right)} \sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi) \\ & \times \left\{\frac{4n^2 - 1^2}{1\,!\,8x} - \frac{(4n^2 - 1^2)(4n^2 - 3^2)(4n^2 - 5^2)}{3\,!\,(8x)^3} + \ldots\right\}; \end{aligned} \qquad (57)$$

while, since $J_n(x) = e^{in\pi} J_n(xe^{-i\pi})$, the asymptotic expansion when $x \neq 0$, $0 < \operatorname{amp} x < 2\pi$, is

$$\begin{aligned} J_n(x) = {} & ie^{in\pi}\sqrt{\left(\frac{2}{\pi x}\right)} \cos(x + \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi) \\ & \times \left\{1 - \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\,(8x)^2} + \ldots\right\} \\ & - ie^{in\pi}\sqrt{\left(\frac{2}{\pi x}\right)} \sin(x + \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi) \\ & \times \left\{\frac{4n^2 - 1^2}{1\,!\,8x} - \frac{(4n^2 - 1^2)(4n^2 - 3^2)(4n^2 - 5^2)}{3\,!\,(8x)^3} + \ldots\right\}. \end{aligned} \qquad (58)$$

If n is half an odd integer, the series in (57) terminates, and gives the complete expressions for the functions discussed in Chapter XIV., § 4.

By means of the formula

$$\pi i I_n(x) = e^{-in\pi} K_n(x) - K_n(xe^{i\pi}) \qquad (59)$$

it can be deduced from (53) that $I_n(x)$ has the asymptotic expansion

$$\begin{aligned} I_n(x) = {} & e^{-i(n + \frac{1}{2}\pi)} \frac{1}{\sqrt{(2\pi x)}} e^{-x} \\ & \times \left\{1 + \frac{4n^2 - 1^2}{1\,!\,8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\,(8x)^2} + \ldots\right\} \\ & + \frac{1}{\sqrt{(2\pi x)}} e^{x} \left\{1 - \frac{4n^2 - 1^2}{1\,!\,8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\,(8x)^2} - \ldots\right\}, \end{aligned} \qquad (60)$$

if $x \neq 0$, $-\frac{3}{2}\pi < \operatorname{amp} x < \frac{1}{2}\pi$; while from the formula

$$I_n(x) = e^{n\pi i} I_n(xe^{-i\pi}),$$

it follows that the expansion when $x \neq 0$, $-\frac{1}{2}\pi < \operatorname{amp} x < \frac{3}{2}\pi$ is

$$\begin{aligned}
\mathrm{I}_n(x) = {} & \frac{1}{\sqrt{(2\pi x)}} e^{x}\left\{1 - \frac{4n^2 - 1^2}{1\,!\; 8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\;(8x)^2} - \dots\right\} \\
& + e^{(n+\frac{1}{2})\pi i}\frac{1}{\sqrt{(2\pi x)}} e^{-x} \\
& \times \left\{1 + \frac{4n^2 - 1^2}{1\,!\; 8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2\,!\;(8x)^2} + \dots\right\}. \qquad (61)
\end{aligned}$$

§ 4. **Lommel Integrals.** If u and v are solutions of the equations

$$x^2u'' + xu' + (\lambda^2x^2 - m^2)u = 0, \quad . \quad . \quad (62)$$

$$x^2v'' + xv' + (\mu^2x^2 - n^2)v = 0, \quad . \quad . \quad (63)$$

then, multiplying (62) and (63) by v/x and u/x respectively, and subtracting, we find that

$$\frac{d}{dx}\{x(uv' - vu')\} = \left\{(\lambda^2 - \mu^2)x + \frac{n^2 - m^2}{x}\right\}uv,$$

and that therefore

$$\int_a^b\left\{(\lambda^2 - \mu^2)x + \frac{n^2 - m^2}{x}\right\}uv\,dx = \Big[x(uv' - vu')\Big]_a^b. \quad (64)$$

In particular, suppose that $m = n$, and let $u = \mathrm{J}_n(\lambda x)$, $v = \mathrm{J}_n(\mu x)$; then, if $n > -1$,

$$\begin{aligned}
(\lambda^2 - \mu^2)&\int_0^x x\mathrm{J}_n(\lambda x)\mathrm{J}_n(\mu x)dx \\
&= x\{\mu\mathrm{J}_n(\lambda x)\mathrm{J}_n'(\mu x) - \lambda\mathrm{J}_n(\mu x)\mathrm{J}_n'(\lambda x)\} \quad . \quad . \quad (65) \\
&= -x\{\mu\mathrm{J}_n(\lambda x)\mathrm{J}_{n+1}(\mu x) - \lambda\mathrm{J}_n(\mu x)\mathrm{J}_{n+1}(\lambda x)\}, \quad . \quad (66)
\end{aligned}$$

from (XIV., 31).

In (65) put $\mu = \lambda + \epsilon$, where ϵ is small; then

$$\begin{aligned}
(-2\lambda\epsilon - \epsilon^2)&\int_0^x x\mathrm{J}_n(\lambda x)\{\mathrm{J}_n(\lambda x) + \epsilon x\mathrm{J}_n'(\lambda x) + \dots\}dx \\
&= x\left\{\begin{matrix}\epsilon\mathrm{J}_n(\lambda x)\mathrm{J}_n'(\lambda x) + \epsilon\lambda x\mathrm{J}_n(\lambda x)\mathrm{J}_n''(\lambda x) \\ - \epsilon\lambda x\mathrm{J}_n'(\lambda x)\mathrm{J}_n'(\lambda x) + \epsilon^2(\dots)\end{matrix}\right\},
\end{aligned}$$

so that, if we divide by ϵ, and make ϵ tend to zero, we get $(n > -1)$

$$\begin{aligned}
\int_0^x x\{\mathrm{J}_n(\lambda x)\}^2dx &= \frac{x^2}{2}\left[\{\mathrm{J}_n'(\lambda x)\}^2 - \mathrm{J}_n(\lambda x)\mathrm{J}_n''(\lambda x) - \frac{1}{\lambda x}\mathrm{J}_n(\lambda x)\mathrm{J}_n'(\lambda x)\right] \\
&= \frac{x^2}{2}\left[\{\mathrm{J}_n'(\lambda x)\}^2 + \left(1 - \frac{n^2}{\lambda^2x^2}\right)\{\mathrm{J}_n(\lambda x)\}^2\right]. \quad (67)
\end{aligned}$$

Again, in (64) let $\mu = \lambda$ and $u = J_m(\lambda x)$, $v = J_n(\lambda x)$; then, if $m + n > 0$, and $m \neq n$,

$$\int_0^1 J_m(\lambda x)J_n(\lambda x)\frac{dx}{x} = \frac{\lambda}{m^2 - n^2}\{J_n(\lambda)J'_m(\lambda) - J_m(\lambda)J'_n(\lambda)\}, \quad (68)$$

and, if $n > 0$,

$$\int_0^1 \{J_n(\lambda x)\}^2\frac{dx}{x} = \frac{\lambda}{2n}\left\{J_n(\lambda)\frac{\partial}{\partial n}J'_n(\lambda) - J'_n(\lambda)\frac{\partial}{\partial n}J_n(\lambda)\right\}. \quad (69)$$

In the same way it can be deduced from equation (12) that, if $n > -1$,

$$(\lambda^2 - \mu^2)\int_0^x xI_n(\lambda x)I_n(\mu x)dx = x\{\lambda I_n(\mu x)I'_n(\lambda x) - \mu I_n(\lambda x)I'_n(\mu x)\}, \quad (70)$$

and

$$\int_0^x x\{I_n(\lambda x)\}^2dx = -\frac{x^2}{2}\left[\{I'_n(\lambda x)\}^2 - \left(1 + \frac{n^2}{\lambda^2x^2}\right)\{I_n(\lambda x)\}^2\right]; \quad (71)$$

while, if $R(\lambda + \mu) > 0$,

$$(\lambda^2 - \mu^2)\int_x^\infty xK_n(\lambda x)K_n(\mu x)dx = x\{\mu K_n(\lambda x)K'_n(\mu x) - \lambda K_n(\mu x)K'_n(\lambda x)\}, \quad (72)$$

and, if $R(\lambda) > 0$,

$$\int_x^\infty x\{K_n(\lambda x)\}^2dx = \frac{x^2}{2}\left[\{K'_n(\lambda x)\}^2 - \left(1 + \frac{n^2}{\lambda^2x^2}\right)\{K_n(\lambda x)\}^2\right]. \quad (73)$$

Formulæ corresponding to (68) and (69) can also be deduced from (12).

Again, from (64), if

$$u = J_n(ax)G_n(\rho x) - J_n(\rho x)G_n(ax),$$
$$v = J_n(ay)G_n(\rho y) - J_n(\rho y)G_n(ay),$$

where a and b are real and positive, and u and v are regarded as functions of ρ,

$$\int_a^b \rho uv\,d\rho = \frac{1}{x^2 - y^2}\left\{\rho\left(u\frac{\partial v}{\partial \rho} - v\frac{\partial u}{\partial \rho}\right)\right\}_{\rho = b}, \quad (74)$$

and

$$\int_a^b \rho u^2 d\rho = -\frac{1}{2x}\left\{\rho\left(u\frac{\partial^2 u}{\partial\rho\partial x} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial \rho}\right)\right\}_{\rho = b}. \quad (75)$$

Similarly, from (12), if a and b are real and positive, and

$$u = \mathrm{I}_n(ax)\mathrm{K}_n(\rho x) - \mathrm{I}_n(\rho x)\mathrm{K}_n(ax),$$
$$v = \mathrm{I}_n(ay)\mathrm{K}_n(\rho y) - \mathrm{I}_n(\rho y)\mathrm{K}_n(ay),$$
$$\int_a^b \rho uv d\rho = -\frac{1}{x^2 - y^2}\left\{\rho\left(u\frac{\partial v}{\partial \rho} - v\frac{\partial u}{\partial \rho}\right)\right\}_{\rho = b}, \quad . \quad (76)$$

and

$$\int_a^b \rho u^2 d\rho = \frac{1}{2x}\left\{\rho\left(u\frac{\partial^2 u}{\partial\rho\partial x} - \frac{\partial u}{\partial x}\,\frac{\partial u}{\partial \rho}\right)\right\}_{\rho = b}. \quad . \quad (77)$$

§ 5. **Zeros of the Bessel Functions.** In this section a number of theorems regarding the zeros of the Bessel Functions will be given.

Theorem I. All the zeros of any Bessel Function whose argument is x are simple zeros, except possibly $x = 0$.

This follows from Bessel's Equation and the theorem of Chapter V., § 5.

Corollary 1. $\mathrm{J}_n(x)$ and $\mathrm{J}_n'(x)$ have no common zeros, except possibly $x = 0$. This is also true of any other Bessel Function.

Corollary 2. The functions $\mathrm{J}_n'(x)$ and $ax\mathrm{J}_n'(x) + b\mathrm{J}_n(x)$ have no repeated zeros except possibly $x = 0$.

This follows because these functions satisfy the equations

$$x^2(x^2 - n^2)y'' + x(x^2 - 3n^2)y' + \{(x^2 - n^2)^2 - (x^2 + n^2)\}y = 0,$$
$$x^2\{a^2(x^2 - n^2) + b^2\}y'' - x\{a^2(x^2 + n^2) - b^2\}y'$$
$$+ \{a^2(x^2 - n^2)^2 + 2abx^2 + b^2(x^2 - n^2)\}y = 0,$$

respectively.

Theorem II. Two linearly independent solutions, P and Q, of Bessel's Equation cannot have any common zeros except possibly $x = 0$.

For, from (XIV., 74),

$$\mathrm{PQ}' - \mathrm{P}'\mathrm{Q} = \frac{\mathrm{C}}{x},$$

and, if P and Q both vanished for the same value of x, C would necessarily be zero: it would then follow that

$$\mathrm{P}'/\mathrm{P} = \mathrm{Q}'/\mathrm{Q},$$

so that P would be a constant multiple of Q; but this is impossible, since P and Q are linearly independent.

For instance, $J_n(x)$ and $Y_n(x)$ cannot have any common zeros.

Theorem III. Between any two positive or negative zeros of any real Bessel Function of order n one and only one zero of any other real Bessel Function of order n will lie.

This is a particular case of the following general theorem.

Let y_1 and y_2 be independent real integrals of the equation

$$ay'' + by' + cy = 0,$$

where the coefficients a, b, c are real continuous functions of x in the interval (α, β); then, if a does not vanish in this interval, and if x_1 and x_2 are two consecutive zeros of y_1 within the interval, there will be one and only one zero of y_2 within the interval (x_1, x_2).

For, let $y_2 = vy_1$; then, differentiating and substituting in the differential equation, we get

$$2av'y_1' + av''y_1 + bv'y_1 = 0,$$

so that

$$\frac{v''}{v'} = -\frac{b}{a} - 2\frac{y_1'}{y_1},$$

and therefore

$$v' = \frac{C}{y_1^2} e^{-\int_{x_1}^{x} \frac{b}{a} dx},$$

where C is a constant.

It follows that v' is always positive or always negative as x increases from x_1 to x_2. But, when x has the value x_1 or x_2, v is infinite: hence, as x increases from x_1 to x_2, v either increases from $-\infty$ to $+\infty$, or decreases from $+\infty$ to $-\infty$. Thus v and therefore y_2 vanishes once and only once in the interval (x_1, x_2).

For example, between any two consecutive positive or negative zeros of $J_n(x)$ there is one and only one zero of $Y_n(x)$.

Theorem IV. The functions J_n and J_{n+1} cannot have a common zero, except possibly $x = 0$.

This follows from the formula (XIV., 31)

$$J_n' - nJ_n/x = -J_{n+1};$$

for, if J_n and J_{n+1} had a common zero, it would be a common zero of J_n and J'_n, which, by *Theorem I.*, *Cor.* 1, is impossible. From the corresponding formulæ for the other Bessel and Modified Bessel Functions like theorems can be deduced.

Theorem V. Between any two consecutive real zeros of $x^{-n}J_n$ there lies one and only one zero of $x^{-n}J_{n+1}$.

For, (XIV., 37),

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1},$$

and therefore, since $x^{-n}J_n$ and $x^{-n}J_{n+1}$ are continuous functions, it follows from Rolle's Theorem that between each two consecutive zeros of $x^{-n}J_n$ there lies at least one zero of $x^{-n}J_{n+1}$. In the same way, from the formula (XIV., 36)

$$\frac{d}{dx}(x^{n+1}J_{n+1}) = x^{n+1}J_n,$$

it follows that, between each successive pair of zeros of $x^{n+1}J_{n+1}$ there lies at least one zero of $x^{n+1}J_n$.

Thus the theorem is proved, except for the interval $(-k, k)$, where k is the smallest positive zero of $x^{-n}J_n$. But $x^{-n}J_{n+1}$ has a simple zero at $x = 0$, and, if there were another zero λ of $x^{-n}J_{n+1}$ between 0 and k, there would be another zero of $x^{-n}J_n$ between λ and k, which, of course, is impossible. Thus the theorem is proved.

Theorem VI. If $n > -1$, $J_n(x)$ cannot have any complex zeros.

For, if $p + iq$ were a zero, where p and q are real, $p - iq$ would also be a zero, and therefore, from (65),

$$\int_0^1 xJ_n\{(p + iq)x\}J_n\{(p - iq)x\}dx = 0.$$

But the second and third factors of the integrand are conjugate complex numbers; hence the integrand, and consequently the integral must be positive, which cannot be the case. Thus the theorem is true.

Similarly, from (72) it can be deduced that $K_n(x)$ cannot have any complex zeros whose real parts are positive.

Theorem VII. If $n > -1$, $J_n(x)$ cannot have any purely imaginary zeros.

For a purely imaginary zero of $J_n(x)$ would be a real zero of $I_n(x)$; and, from (15), it is evident that, when x is real, every term in the series for $x^{-n}I_n(x)$ is positive, so that $I_n(x)$ cannot vanish.

Theorem VIII. $K_n(x)$ has no real positive zeros.

This follows from (47) and (17).

Theorem IX. $G_n(x)$ has no real zeros.

For, if $G_n(x)$ were zero, the real and imaginary parts of $G_n(x)$ would both be zero: but, from (XIV., 69), it follows that $J_n(x)$ and $Y_n(x)$ would then both vanish, which, by *Theorem II.*, is impossible.

Corollary. $K_n(x)$ cannot have a purely imaginary zero.

Theorem X. The function $J_n(ax)G_n(bx) - J_n(bx)G_n(ax)$, where a and b are real and positive, is a uniform, even function of x, whose zeros are all real and simple.

From (XIV., 67) it follows that the function is uniform and even. By putting $x = p + iq, y = p - iq$ in (74), it can be shown, as in *Theorem VI.*, that the function has no complex zeros. Again, suppose that $x = iq$ is a purely imaginary zero; then

$$\frac{I_n(aq)}{I_n(bq)} = \frac{K_n(aq)}{K_n(bq)}.$$

But, from (15) it is clear that, if $n \geqq 0$ and $b > a$, $I_n(aq)/I_n(bq) < 1$; while, from (47), $K_n(aq)/K_n(bq) > 1$. Thus, if $n \geqq 0$, the function cannot have an imaginary zero. But, since $K_{-n} = K_n$, this is also true when n is negative. Finally, the function cannot have a repeated zero, x_1. For, if it had, then in (75) the right-hand side would vanish for $x = x_1$, while the left-hand side is necessarily positive.

Theorem XI. If $n > -1$, $J_n'(x)$ cannot have any complex zeros.

For, if $\lambda = p + iq$ is a zero of $J_n'(x)$, so also is $\mu = p - iq$. Hence, from (65),

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x) = 0,$$

which is impossible, since $J_n(\lambda x)J_n(\mu x)$, being the product of two conjugate complex numbers, is necessarily positive.

This is also true of the function $ax\mathrm{J}_n'(x) + b\mathrm{J}_n(x)$. For, if $\lambda = p + iq$, $\mu = p - iq$ are zeros,

$$a\lambda\mathrm{J}_n'(\lambda) + b\mathrm{J}_n(\lambda) = 0,$$
$$a\mu\mathrm{J}_n'(\mu) + b\mathrm{J}_n(\mu) = 0,$$

and therefore, eliminating a and b, we get

$$\mu\mathrm{J}_n(\lambda)\mathrm{J}_n'(\mu) - \lambda\mathrm{J}_n(\mu)\mathrm{J}_n'(\lambda) = 0,$$

so that, from (65),

$$\int_0^1 x\mathrm{J}_n(\lambda x)\mathrm{J}_n(\mu x)dx = 0,$$

which is impossible.

Theorem XII. If $n \geqq 0$, $\mathrm{J}_n'(x)$ cannot have an imaginary zero.

For, if it had an imaginary zero iq, then $\mathrm{I}_n'(q)$ would vanish. But, from (20),

$$q^{1-n}\mathrm{I}_n'(q) = nq^{-n}\mathrm{I}_n(q) + q^{1-n}\mathrm{I}_{n+1}(q),$$

and, if q is not zero, both terms on the right are positive except when $n = 0$, in which case the first is zero and the second positive: thus $\mathrm{I}_n'(q)$ cannot vanish.

Similarly, if a and b are positive, and $n \geqq 0$, $ax\mathrm{J}_n'(x) + b\mathrm{J}_n(x)$ has no imaginary zeros.

Theorem XIII. If $n > 0$, and if a and b are the least positive zeros of $\mathrm{J}_n(x)$ and $\mathrm{J}_n'(x)$ respectively, then $a > b > n$.

For since, when $n > 0$, $\mathrm{J}_n'(x)$ is positive for x small and positive, $\mathrm{J}_n(b)$, the first turning value of $\mathrm{J}_n(x)$, is a maximum. Hence $\mathrm{J}_n''(b)$ is negative, $\mathrm{J}_n(b)$ is positive, and $\mathrm{J}_n'(b) = 0$. Thus, from Bessel's Equation,

$$b^2\mathrm{J}_n''(b) + (b^2 - n^2)\mathrm{J}_n(b) = 0,$$

and therefore $b^2 - n^2$ must be positive: hence $a > b > n$.

Theorem XIV. If $n \geqq 0$, between two consecutive zeros of $\mathrm{J}_n(x)$ there lies one and only one zero of $\mathrm{J}_n'(x)$.

For, by Rolle's Theorem, there must be at least one such zero of $\mathrm{J}_n'(x)$, and, if there are more, there must be at least three. But, for one at least of the three, b say, $\mathrm{J}_n''(b)$ and $\mathrm{J}_n(b)$ must have the same sign. From Bessel's Equation, however, since $b^2 > n^2$, it follows that this cannot be the case when $n > 0$; and, when $n = 0$, Bessel's Equation gives

$$J_0''(b) + J_0(b) = 0,$$

so that in this case also $J_0''(b)$ and $J_0(b)$ must have unlike signs. Hence the theorem is proved.

Theorem XV. If $n \geqq 0$, between two consecutive positive zeros of $J_n(x)$ there lies one and only one zero of $axJ_n'(x) + bJ_n(x)$.

For, if
$$y = J_n'^2 + \left(1 - \frac{n^2}{x^2}\right)J_n^2,$$
$$\frac{d}{dx}\left(\frac{xJ_n'}{J_n}\right) = -\frac{xy}{J_n^2}.$$

Hence, between any two consecutive positive zeros, p and q, $\frac{d}{dx}\left(\frac{xJ_n'}{J_n}\right)$ is negative, since $q > p > n$, and therefore xJ_n'/J_n steadily decreases from $+\infty$ to $-\infty$ in this interval. Thus $axJ_n'(x) + bJ_n(x)$ has one and only one zero in the interval.

In the interval $(0, \beta)$, where β is the first positive zero of $J_n(x)$, there lies either one or no zero of $axJ_n'(x) + bJ_n(x)$. For

$$\frac{d}{dx}(x^2y) = 2xJ_n^2,$$

and therefore, since x^2y is zero when $x = 0$, it is positive when x is positive. Thus xJ_n'/J_n decreases steadily in the interval $(0, \beta)$. But $\lim\limits_{x \to 0}(xJ_n'/J_n) = n$; hence xJ_n'/J_n decreases steadily from n to $-\infty$ as x increases from 0 to β.

Theorem XVI. The function $J_n(x)$ has an infinite number of real zeros.

This follows from the formula (57); for, when x is large, the sign of $J_n(x)$ is, as a rule, determined by that of the term

$$\sqrt{\left(\frac{2}{\pi x}\right)} \cos\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right),$$

and this term changes sign when x increases by π, so that there must be one zero in each interval of length π. In fact, when x is large, the zeros are given approximately by

$$x = \left(r + \tfrac{3}{4} + \tfrac{1}{2}n\right)\pi,$$

where r is a large positive integer. To each positive root there corresponds a numerically equal negative root.

It follows that each of the functions Y_n, J'_n, $axJ'_n + bJ_n$, has an infinite number of real zeros.

§ 6. **The Fourier-Bessel Expansions.** If we assume that a function $f(x)$, which satisfies Dirichlet's Conditions in the interval (0, a) can be expanded in a series

$$f(x) = \sum_{r=1}^{\infty} A_r J_n(\lambda_r x), \quad . \quad . \quad . \quad (78)$$

where $n > -1$ and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_n(\lambda a)$ taken in order, and if we further assume that the series in (78) can be integrated term by term over the range (0, a), the coefficients can be found as follows. Multiply by $xJ_n(\lambda_r x)$, and integrate over this range; then from (65) we see that all the integrals on the right except the rth will vanish, and from (67) we deduce that

$$\int_0^a xf(x)J_n(\lambda_r x)dx = \frac{a^2}{2}A_r\{J'_n(\lambda_r a)\}^2. \quad . \quad (79)$$

From this equation the value of A_r is obtained. A proof of the validity of the expansion lies beyond the scope of the present volume, but it can be assumed that, if $0 < x < a$, the series on the right of (78) has the value

$$\tfrac{1}{2}\{f(x + 0) + f(x - 0)\}.$$

When $x = a$, the value of the series is obviously zero.

When in (78) $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of

$$\lambda J'_n(\lambda a) + hJ_n(\lambda a),$$

it may be deduced from (65) (cf. § 5, *Theorem XI.*) and from (67) that

$$\int_0^a xf(x)J_n(\lambda_r x)dx = \frac{a^2}{2}A_r\frac{a^2h^2 + (a^2\lambda_r^2 - n^2)}{a^2\lambda_r^2}\{J_n(\lambda_r a)\}^2; \quad (80)$$

and, in particular, when h is zero, the λ's are the zeros of $J'_n(\lambda a)$, and

$$\int_0^a xf(x)J_n(\lambda_r x)dx = \frac{a^2}{2}A_r\left(1 - \frac{n^2}{a^2\lambda_r^2}\right)\{J_n(\lambda_r a)\}^2. \quad (81)$$

These expansions are valid for $0 < x < a$.

Another Fourier-Bessel expansion is

$$f(x) = \sum_{r=1}^{\infty} A_r\{J_n(\lambda_r x)G_n(\lambda_r a) - G_n(\lambda_r x)J_n(\lambda_r a)\}, \qquad (82)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of

$$J_n(\lambda b)G_n(\lambda a) - G_n(\lambda b)J_n(\lambda a),$$

regarded as a function of λ (§ 5, *Theorem X.*). Here, from (74) and (75),

$$\begin{aligned}\int_a^b & xf(x)\{J_n(\lambda_r x)G_n(\lambda_r a) - G_n(\lambda_r x)J_n(\lambda_r a)\}dx \\ &= \frac{b}{2}A_r\frac{\partial}{\partial\lambda_r}\{J_n(\lambda_r b)G_n(\lambda_r a) - G_n(\lambda_r b)J_n(\lambda_r a)\} \\ &\qquad \times \{J_n'(\lambda_r b)G_n(\lambda_r a) - G_n'(\lambda_r b)J_n(\lambda_r a)\}. \qquad (83)\end{aligned}$$

Now
$$\frac{G_n(\lambda_r a)}{G_n(\lambda_r b)} = \frac{J_n(\lambda_r a)}{J_n(\lambda_r b)} = \beta, \text{ say.}$$

Hence
$$\begin{aligned}&\frac{\partial}{\partial\lambda_r}\{J_n(\lambda_r b)G_n(\lambda_r a) - G_n(\lambda_r b)J_n(\lambda_r a)\} \\ &= b\{J_n'(\lambda_r b)G_n(\lambda_r a) - G_n'(\lambda_r b)J_n(\lambda_r a)\} \\ &\quad + a\{J_n(\lambda_r b)G_n'(\lambda_r a) - G_n(\lambda_r b)J_n'(\lambda_r a)\} \\ &= b\beta\{J_n'(\lambda_r b)G_n(\lambda_r b) - G_n'(\lambda_r b)J_n(\lambda_r b)\} \\ &\quad + \frac{a}{\beta}\{J_n(\lambda_r a)G_n'(\lambda_r a) - G_n(\lambda_r a)J_n'(\lambda_r a)\} \\ &= \frac{\beta}{\lambda_r} - \frac{1}{\lambda_r\beta}, \text{ by (XIV., 76).}\end{aligned}$$

Similarly

$$J_n'(\lambda_r b)G_n(\lambda_r a) - G_n'(\lambda_r b)J_n(\lambda_r a) = \beta/(\lambda_r b).$$

Hence (83) can be written

$$\int_a^b xf(x)\{J_n(\lambda_r x)G_n(\lambda_r a) - G_n(\lambda_r x)J_n(\lambda_r a)\}dx = A_r\frac{J_n^2(\lambda_r a) - J_n^2(\lambda_r b)}{2\lambda_r^2 J_n^2(\lambda_r b)}. \qquad (84)$$

The expansion holds if $a < x < b$. When x is equal to a or b the value of the series (82) is obviously zero.

Examples.

1. Prove that

(i) $t^2 I''_n = (n^2 - n + t^2)I_n - tI_{n+1}$,

(ii) $t^2 K''_n = (n^2 - n + t^2)K_n + tK_{n+1}$.

2. Show that

(i) $G_{\frac{1}{2}}(x) = \sqrt{\left(\frac{\pi}{2x}\right)}e^{ix}$, (ii) $G_{-\frac{1}{2}}(x) = i\sqrt{\left(\frac{\pi}{2x}\right)}e^{ix}$,

(iii) $I_{\frac{1}{2}}(t) = \sqrt{\left(\frac{2}{\pi t}\right)}\sinh t$, (iv) $I_{-\frac{1}{2}}(t) = \sqrt{\left(\frac{2}{\pi t}\right)}\cosh t$,

(v) $K_{\frac{1}{2}}(t) = \sqrt{\left(\frac{\pi}{2t}\right)}e^{-t}$, (vi) $K_{-\frac{1}{2}}(t) = \sqrt{\left(\frac{\pi}{2t}\right)}e^{-t}$.

3. If x and u are real, prove that

$$\int_0^\infty J_1(xu)dx = \frac{1}{u}.$$

[Use XIV., 34.]

4. If x is real, and $n > \frac{1}{2}$, prove that

$$\int_0^\infty x^{1-n}J_n(x)dx = 2^{1-n}/\Gamma(n).$$

[Use XIV., 37.]

5. If a and b are positive constants, prove that

$$\int_0^\infty \sin(ax)J_0(bx)\frac{dx}{x} = \begin{cases}\frac{1}{2}\pi, \text{ if } a \geq b,\\ \sin^{-1}\left(\frac{a}{b}\right), \text{ if } a \leq b.\end{cases}$$

[Apply (XIV., 48), and change the order of integration.]

6. If a and b are positive constants, prove that

$$\int_0^\infty \cos(ax)J_0(bx)dx = \begin{cases}0, \text{ if } a > b,\\ \frac{1}{\sqrt{(b^2 - a^2)}}, \text{ if } a < b.\end{cases}$$

[Differentiate with regard to a in *ex.* 5.]

7. If $R(b \pm ia) > 0$, show that

$$\int_0^\infty e^{-bx}J_0(ax)dx = \frac{1}{\sqrt{(a^2 + b^2)}}.$$

8. If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_0(x)$ taken in order, show that, for $0 < x < 1$,

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}.$$

9. Show that

$$(n^2 - m^2)\int_x^\infty J_n(x)J_m(x)\frac{dx}{x}$$
$$= \frac{2}{\pi}\sin(n - m)\frac{\pi}{2} - xJ_m(x)J'_n(x) + xJ'_m(x)J_n(x).$$

[Use *ex.* 15, Chapter XIV.]

CHAPTER XVI

APPLICATIONS OF BESSEL FUNCTIONS

§ 1. **Vibrations of a Circular Membrane.** In dealing with the vibrations of a circular membrane, such as a drum-head, we transform the equation (III., 42) to polar co-ordinates by means of the equations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and the resulting equation is

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \quad . \quad . \quad (1)$$

Here put $z = \mathrm{T}\mathrm{R}\Theta$, where T, R, and Θ are functions of t, r, and θ alone, and divide by $\mathrm{T}\mathrm{R}\Theta$; the resulting equation is

$$\frac{\dfrac{d^2\mathrm{T}}{dt^2}}{\mathrm{T}} = \frac{c^2}{\mathrm{R}}\left(\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr}\right) + \frac{c^2}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}. \quad . \quad (2)$$

In this equation the last term must be independent of θ, and, in consequence, the value of $\dfrac{d^2\Theta}{d\theta^2}\Big/\Theta$ is constant. But Θ must be periodic, of period 2π, in θ; hence

$$\frac{d^2\Theta}{d\theta^2} = -n^2\Theta, \quad . \quad . \quad . \quad . \quad (3)$$

where n is an integer, and

$$\Theta = \mathrm{A}\cos n\theta + \mathrm{B}\sin n\theta. \quad . \quad . \quad (4)$$

Again, the expression on the left of (2) is constant. But for a normal mode of vibration we may assume, as in the case of a stretched string or rectangular membrane, that

$$\mathrm{T} = \mathrm{C}\cos pt + \mathrm{D}\sin pt = \mathrm{K}\cos(pt + \mathrm{E}); \quad (5)$$

hence the constant is $-p^2$, and

$$\frac{d^2\mathrm{T}}{dt^2} = -p^2\mathrm{T}. \quad . \quad . \quad . \quad . \quad (6)$$

Accordingly, from (2), (3), and (6) we deduce that

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 - n^2)R = 0, \quad . \quad . \quad (7)$$

where $k = p/c$. The solution of this equation is

$$R = FJ_n(kr) + GY_n(kr), \quad . \quad . \quad . \quad (8)$$

and consequently a solution of (1) is

$$z = (A \cos n\theta + B \sin n\theta)(C \cos pt + D \sin pt)\left\{\begin{matrix} FJ_n(kr) \\ + GY_n(kr) \end{matrix}\right\}, \quad (9)$$

where n is an integer, and $p = kc$.

For a membrane bounded by the fixed circle $r = a$, G must be zero, since $Y_n(kr)$ is infinite when $r = 0$; also $J_n(ka)$ must vanish for all values of θ and t: thus

$$z = (A \cos n\theta + B \sin n\theta)(C \cos pt + D \sin pt)J_n(kr), \quad (10)$$

where $J_n(ka) = 0$.

Again, if the vibrations are symmetrical, and therefore independent of θ, n is zero, and

$$z = (C \cos pt + D \sin pt)J_0(kr), \quad . \quad . \quad (11)$$

where $p = kc$ and $J_0(ka) = 0$.

If k_1, k_2, k_3, . . . are the positive zeros of $J_0(ka)$ taken in order, the normal vibrations of the membrane are obtained by substituting their values for k in (11). Since $J_0(k_mr)$ vanishes for $r = k_1a/k_m$, k_2a/k_m, . . ., $k_{m-1}a/k_m$, the mth mode will have $m - 1$ nodal circles, in addition to the boundary $r = a$.

If the membrane starts from rest, $\frac{\partial z}{\partial t} = 0$ initially, and therefore D must vanish. If, moreover, the initial displacement of the membrane is given by $z = f(r)$, we can assume that

$$z = \sum_{m=1}^{\infty} C_m \cos(p_mt)J_0(k_mr), \quad . \quad . \quad (12)$$

where $p_m = ck_m$; thus, when $t = 0$,

$$f(r) = \sum_{m=1}^{\infty} C_mJ_0(k_mr), \quad . \quad . \quad . \quad (13)$$

and the coefficients are given by (XV., 78, 79). All the conditions of the problem are then satisfied by (12).

In the next place, when the vibrations are not symmetrical, let the membrane start from rest at $t = 0$, and let $z = f(r, \theta)$ initially; then we can assume that

$$z = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{A_{n,m} \cos n\theta + B_{n,m} \sin n\theta\} \cos(p_m t) J_n(k_m r), \quad (14)$$

where $k_1, k_2, k_3, \ldots$ are the positive zeros of $J_n(ka)$ taken in order, and $p_m = ck_m$. When $t = 0$,

$$f(r, \theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{A_{n,m} \cos n\theta + B_{n,m} \sin n\theta\} J_n(k_m r), \quad (15)$$

and therefore, from (I., 2) and (XV., 79),

$$A_{n,m} = \frac{2\int_0^{2\pi}\int_0^a f(r, \theta) J_n(k_m r) r dr \cos n\theta d\theta}{\pi a^2 \{J_n'(k_m a)\}^2}, \quad . \quad (16)$$

where, when $n = 0$, π is replaced by 2π. For $B_{n,m}$ the same formula holds with $\sin n\theta$ in place of $\cos n\theta$.

For an annular membrane bounded by the circles $r = a$, $r = b$, where $a < b$, the solution is of the form

$$z = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{A_{n,m} \cos n\theta + B_{n,m} \sin n\theta\} \cos(p_m t)$$
$$\times \{J_n(k_m r) G_n(k_m a) - G_n(k_m r) J_n(k_m a)\}, \quad (17)$$

where $k_1, k_2, k_3, \ldots$ are the positive zeros, taken in order, of

$$J_n(kb) G_n(ka) - G_n(kb) J_n(ka),$$

regarded as a function of k, and $p_m = ck_m$. The values of the coefficients can be found by means of (XV., 84).

§ 2. **Flow of Heat in a Circular Cylinder.** If the equation of conduction of heat (II., 7) be transformed to cylindrical co-ordinates by means of the equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

it becomes

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}\right). \quad . \quad (18)$$

This form of the equation is suitable for the study of the flow of heat in a circular cylinder with its axis along the z-axis.

Case I. Variable Temperature in an Infinite Cylinder. Let the surface $r = a$ of the cylinder be kept at temperature zero and let the initial temperature be $v = f(r)$. The temperature is then independent of θ and z, and the equation of conduction becomes

$$\frac{1}{\kappa}\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} \quad . \quad . \quad . \quad (19)$$

Here put $v = \mathrm{TR}$, where T and R are functions of t and r alone, and divide by TR; thus

$$\frac{1}{\kappa \mathrm{T}}\frac{d\mathrm{T}}{dt} = \frac{1}{\mathrm{R}}\left(\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr}\right). \quad . \quad . \quad (20)$$

In this equation each side has a constant value, C say; so that

$$\frac{d\mathrm{T}}{dt} = \kappa \mathrm{CT},$$

and therefore $\mathrm{T} = \mathrm{A}e^{\kappa \mathrm{C}t}$. Now, from the nature of the problem, it is clear that T will tend steadily to zero as t increases, and therefore C must be negative. Put $\mathrm{C} = -\lambda^2$, and we have

$$\mathrm{T} = \mathrm{A}e^{-\kappa\lambda^2 t}; \quad . \quad . \quad . \quad (21)$$

while, from (20),

$$\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr} + \lambda^2\mathrm{R} = 0, \quad . \quad . \quad (22)$$

of which the general solution is

$$\mathrm{R} = \mathrm{B}\mathrm{J}_0(\lambda r) + \mathrm{D}\mathrm{Y}_0(\lambda r).$$

But, since $\mathrm{Y}_0(\lambda r)$ is infinite when r is zero, D must be zero; hence

$$v = \mathrm{A}e^{-\kappa\lambda^2 t}\mathrm{J}_0(\lambda r), \quad . \quad . \quad . \quad (23)$$

where, from the surface condition, λ must be a root of the equation $\mathrm{J}_0(\lambda a) = 0$.

The most general solution obtained in this way is

$$v = \sum_{n=1}^{\infty} \mathrm{A}_n e^{-\kappa\lambda_n^2 t}\mathrm{J}_0(\lambda_n r), \quad . \quad . \quad (24)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_0(\lambda a)$ taken in order. When $t = 0$ this gives

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r), \quad . \quad . \quad . \quad (25)$$

and hence, by means of (XV., 79), the values of the coefficients in (24) can be determined, provided that $f(r)$ satisfies Dirichlet's Conditions for $0 \leqq r \leqq a$. Thus, finally,

$$v = \frac{2}{a^2} \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t} J_0(\lambda_n r) \frac{\int_0^a x f(x) J_0(\lambda_n x)\, dx}{\{J_1(\lambda_n a)\}^2}. \quad . \quad . \quad (26)$$

In the next place, suppose that the surface is impervious to heat, and that $v = f(r)$ initially; then in (23) $\frac{\partial v}{\partial r} = 0$ when $r = a$, so that $J_0'(\lambda a) = 0$. Thus v is given by (24) and (25) when $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_0'(\lambda a)$: hence, from (XV., 81),

$$v = \frac{2}{a^2} \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t} J_0(\lambda_n r) \frac{\int_0^a x f(x) J_0(\lambda_n x)\, dx}{\{J_0(\lambda_n a)\}^2}$$

Again, if there is radiation at the surface $r = a$ into a medium at temperature zero, the surface condition is $\frac{\partial v}{\partial r} + hv = 0$, and v is given by (24) and (25), where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are [illegible] positive zeros of $\lambda J_0'(\lambda a) + hJ_0(\lambda a)$ taken in order. Hence, from (XV., 80),

$$v = \frac{2}{a^2} \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t} J_0(\lambda_n r) \frac{\lambda_n^2 \int_0^a x f(x) J_0(\lambda_n x)\, dx}{(h^2 + \lambda_n^2)\{J_0(\lambda_n a)\}^2}. \quad (28)$$

Case II. Steady Temperature in a Semi-Infinite Cylinder. We take the axis of the cylinder as the z-axis, and its base as the plane $z = 0$. The base is kept at steady temperature $v = f(r)$, and radiation takes place at the surface $r = a$ into a medium at temperature zero, so that v is independent of θ. Then equation (18) reduces to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = 0, \quad . \quad . \quad . \quad (29)$$

and the surface conditions are

$$v = f(r), \text{ when } z = 0, \quad . \quad . \quad . \quad (30)$$

$$\frac{\partial v}{\partial r} + hv = 0, \text{ when } r = a. \quad . \quad . \quad (31)$$

If in (29) we put $v = \mathrm{RZ}$, where R and Z are functions of r and z alone, and divide by RZ, we get

$$\frac{1}{\mathrm{R}}\left\{\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr}\right\} + \frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} = 0, \quad . \quad . \quad (32)$$

so that $\frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2}$ has a constant value C, and therefore

$$\mathrm{Z} = \mathrm{A}e^{\sqrt{\mathrm{C}}.z} + \mathrm{B}e^{-\sqrt{\mathrm{C}}.z}.$$

But, since Z tends to zero as z tends to infinity, we take A to [illegible] zero and C to be positive; thus, if $\mathrm{C} = \lambda^2$, $\mathrm{Z} = e^{-\lambda z}$, where [illegible] positive, and (32) becomes

$$\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr} + \lambda^2\mathrm{R} = 0,$$

[illegible]

$$\mathrm{R} = \mathrm{DJ}_0(\lambda r) + \mathrm{EY}_0(\lambda r).$$

Now, since $\mathrm{Y}_0(\lambda r)$ is infinite when r is zero, E must be zero; hence

$$v = e^{-\lambda z}\mathrm{J}_0(\lambda r).$$

Also, from (31),

$$\lambda \mathrm{J}_0'(\lambda a) + h\mathrm{J}_0(\lambda a) = 0 : \quad . \quad . \quad . \quad (33)$$

therefore, if $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of (33), a solution of (29) is

$$v = \sum_{m=1}^{\infty} \mathrm{A}_m e^{-\lambda_m z}\mathrm{J}_0(\lambda_m r),$$

and, from (30), when $z = 0$,

$$f(r) = \sum_{m=1}^{\infty} \mathrm{A}_m \mathrm{J}_0(\lambda_m r),$$

so that the coefficients are given by (XV., 80). Thus, finally,

$$v = \frac{2}{a^2}\sum_{m=1}^{\infty} e^{-\lambda_m{}^2} J_0(\lambda_m r)\frac{\lambda_m^2\int_0^a xf(x)J_0(\lambda_m x)dx}{(h^2+\lambda_m^2)\{J_0(\lambda_m a)\}^2}. \qquad (34)$$

§ 3. **Flow of Heat in a Sphere.** If the equation of conduction of heat (II., 7) be transformed to polar co-ordinates (r, θ, ϕ) by means of the equations

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

it becomes

$$\frac{\partial v}{\partial t} = \kappa\left\{\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right\}. \qquad (35)$$

Case I. Steady Temperature in a Hollow Sphere. If the inner and outer surfaces $r = a$, $r = b$ are kept at temperatures v_1, v_2, respectively, the temperature will be independent of t, θ, and ϕ, and (35) becomes

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0.$$

Here put $v = u/r$, and we get

$$\frac{d^2u}{dr^2} = 0,$$

with $u = av_1$, when $r = a$,
$u = bv_2$, when $r = b$.

On solving these equations, we find that

$$v = \frac{v_1 a(b-r) + v_2 b(r-a)}{r(b-a)}. \qquad (36)$$

Case II. Variable Temperature in a Hollow Sphere. Let the surfaces $r = a$, $r = b$ be kept at constant temperatures v and v_2 respectively, and let $v = f(r)$ initially. Then, if $v = u/r$,

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2}, \qquad (37)$$

$u = av_1$, when $r = a$,
$u = bv_2$, when $r = b$,
and $u = rf(r)$, when $t = 0$.

These equations are similar to those of (II., 15) for a finite rod with ends at fixed temperatures, and the solution can be at once deduced from that given in (II., 16). The solution for a solid sphere can then be obtained by putting $a = 0$.

Case III. Unsymmetrical Variable Temperature in a Solid Sphere. Let the surface $r = a$ be kept at temperature zero, and let $v = f(r, \theta, \phi)$ initially. In equation (35) put $v = e^{-\kappa\lambda^2 t}u$ and $\cos\theta = \mu$, and it becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial u}{\partial\mu}\right\} + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 u}{\partial\phi^2} + \lambda^2 u = 0. \quad (38)$$

Comparing this equation with (IV., 11), we see from Ch. IV., § 6, that it will be satisfied by

$$u = \mathrm{R}\{\mathrm{A}\mathrm{T}_n^m(\mu) + \mathrm{B}\mathrm{Q}_n^m(\mu)\}\{\mathrm{C}\cos m\phi + \mathrm{D}\sin m\phi\}, \quad (39)$$

provided that R, a function of r alone, satisfies

$$r^2\frac{d^2\mathrm{R}}{dr^2} + 2r\frac{d\mathrm{R}}{dr} + \{\lambda^2 r^2 - n(n+1)\}\mathrm{R} = 0. \quad . \quad (40)$$

Since u remains finite when $\mu = \pm 1$, B must be zero and n a positive integer [cf Ch VI., *exs.* 9 and 10]. Also, if in (40) we put $\mathrm{R} = r^{-\frac{1}{2}}y$, it is transformed into the equation

$$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} + \{\lambda^2 r^2 - (n+\tfrac{1}{2})^2\}y = 0,$$

of which the solution is

$$y = \mathrm{A}\mathrm{J}_{n+\frac{1}{2}}(\lambda r) + \mathrm{B}\mathrm{J}_{-n-\frac{1}{2}}(\lambda r).$$

Hence

$$\mathrm{R} = \mathrm{A}r^{-\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\lambda r) + \mathrm{B}r^{-\frac{1}{2}}\mathrm{J}_{-n-\frac{1}{2}}(\lambda r).$$

But, since R remains finite when r vanishes, B must be zero; therefore

$$v = e^{-\kappa\lambda^2 t}(\lambda r)^{-\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\lambda r)\mathrm{T}_n^m(\mu)\{\mathrm{A}\cos m\phi + \mathrm{B}\sin m\phi\}.$$

Since v is zero when $r = a$, λ must be a zero of $\mathrm{J}_{n+\frac{1}{2}}(\lambda a)$: hence the complete solution of the problem is

$$v = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{p=1}^{\infty} e^{-\kappa\lambda_p^2 t}(\lambda_p r)^{-\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\lambda_p r)$$
$$\times\ \mathrm{T}_n^m(\mu)\{\mathrm{A}_{n,m,p}\cos m\phi + \mathrm{B}_{n,m,p}\sin m\phi\}, \quad (41)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the zeros of $J_{n+\frac{1}{2}}(\lambda a)$, provided that, when $t = 0$,

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{p=1}^{\infty} (\lambda_p r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_p r) T_n^m(\mu) \times (A_{n,m,p} \cos m\phi + B_{n,m,p} \sin m\phi). \quad (42)$$

From (VII., 16, 18) and (XV., 79) it follows that

$$\int_0^{2\pi} \int_{-1}^{1} \int_0^a f(x, \theta, \phi) x^{\frac{3}{2}} J_{n+\frac{1}{2}}(\lambda_p x) T_n^m(\mu) \cos m\phi \, dx \, d\mu \, d\phi = \frac{\pi a^2 \lambda_p^{-\frac{1}{2}}}{2n+1} \frac{(n+m)!}{(n-m)!} \{J'_{n+\frac{1}{2}}(\lambda_p a)\}^2 A_{n,m,p}, \quad (43)$$

π being replaced by 2π when $m = 0$. In the formula for $B_{n,m,p}$, $\cos m\phi$ is replaced by $\sin m\phi$.

Examples.

1. Show that $e^{-\kappa\lambda^2 t} J_n(\lambda r)\{A \cos n\theta + B \sin n\theta\}$ is a solution of the equation

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right),$$

and deduce that, for an infinite cylinder whose surface $r = a$ is kept at temperature zero, and whose initial temperature is $v = f(r, \theta)$, the solution is

$$v = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (A_{n,m} \cos n\theta + B_{n,m} \sin n\theta) J_n(\lambda_m r) e^{-\kappa\lambda_m^2 t},$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_n(\lambda a)$, and

$$A_{n,m} = \frac{2}{\pi a^2 \{J'_n(\lambda_m a)\}^2} \int_0^{2\pi} \int_0^a f(x, \theta) J_n(\lambda_m x) x \cos n\theta \, dx \, d\theta,$$

the 2 being omitted when $n = 0$, and $\cos n\theta$ being replaced by $\sin n\theta$ in the formula for $B_{n,m}$.

2. If radiation takes place at the surface $r = a$ of an infinite cylinder into a medium at temperature zero, and if $v = f(r, \theta)$ initially, show that

$$v = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (A_{n,m} \cos n\theta + B_{n,m} \sin n\theta) J_n(\lambda_m r) e^{-\kappa\lambda_m^2 t},$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $\lambda J'_m(\lambda a) + h J_m(\lambda a)$

and

$$A_{n,m} = \frac{2\lambda_m^2 \int_0^{2\pi}\int_0^a f(x,\ \theta) J_n(\lambda_m x) x \cos n\theta \, dx d\theta}{\pi a^2 \left(\lambda_m^2 + h^2 - \frac{n^2}{a^2}\right) \{J_n(\lambda_m a)\}^2},$$

the 2 being omitted when $n = 0$, and $\cos n\theta$ being replaced by $\sin n\theta$ in the formula for $B_{n,m}$.

3. A finite cylinder is bounded by $r = a$, $z = \pm l$, and these surfaces are kept at temperature zero. If the initial temperature is $v = f(r,\ \theta,\ z)$, show that

$$v = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} e^{-\kappa\left(\lambda_p^2 + \frac{m^2\pi^2}{4l^2}\right)t} J_n(\lambda_p r) \sin \frac{m\pi}{2l}(z + l)$$
$$\times \{A_{n,m,p} \cos n\theta + B_{n,m,p} \sin n\theta\},$$

where λ_1, λ_2, λ_3, . . are the positive zeros of $J_n(\lambda a)$ and

$$A_{n,m,p} = \frac{2}{\pi a^2 l \{J_n'(\lambda_p a)\}^2}$$
$$\times \int_0^{2\pi}\int_{-l}^{l}\int_0^a f(x,\ \theta,\ z) x J_n(\lambda_p x) \sin \frac{m\pi}{2l}(z + l) \cos n\theta \ dx dz d\theta,$$

the factor 2 being omitted when $n = 0$, and $\cos n\theta$ being replaced by $\sin n\theta$ in the formula for $B_{n,m,p}$.

4. Show that for an infinite hollow cylinder bounded by the surfaces $r = a$, $r = b$, where $a < b$, and with initial temperature $v = f(r)$, the solution is

$$v = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t}\{J_0(\lambda_m r) G_0(\lambda_m a) - G_0(\lambda_m r) J_0(\lambda_m a)\},$$

where λ_1, λ_2, λ_3, . . . are the positive zeros of

$$J_0(\lambda b) G_0(\lambda a) - G_0(\lambda b) J_0(\lambda a)$$

regarded as a function of λ, and A_m is given by (XV., 84) with $n = 0$.

5. For an infinite wedge bounded by the surfaces $r = a$, $\theta = 0$, $\theta = \alpha$ all kept at temperature zero, and with the initial temperature $v = f(r,\ \theta)$, show that

$$v = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m} e^{-\kappa\lambda_m^2 t} \sin \frac{n\pi\theta}{\alpha} J_{\frac{n\pi}{\alpha}}(\lambda_m r),$$

where λ_1, λ_2, λ_3, . . . are the positive zeros of $J_{\frac{n\pi}{\alpha}}(\lambda a)$ and

$$A_{n,m} = \frac{4}{a^2\alpha \left\{J_{\frac{n\pi}{\alpha}}'(\lambda_m a)\right\}^2} \int_0^{\alpha}\int_0^{a} f(x,\ \theta)\, x J_{\frac{n\pi}{\alpha}}(\lambda_m x) \sin \frac{n\pi\theta}{\alpha}\, dx d\theta.$$

6. For a wedge bounded by the surfaces $r = a$, $\theta = 0$, $\theta = \alpha$, $z = \pm l$, all kept at temperature zero, and with the initial temperature $v = f(r, \theta, z)$, show that

$$v = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{n,m,p}\, e^{-\kappa\left(\lambda_p^2 + \frac{m^2\pi^2}{4l^2}\right)t} \sin\frac{m\pi}{2l}(z+l)$$

$$\times \sin\frac{n\pi\theta}{\alpha} J_{\frac{n\pi}{\alpha}}(\lambda_p r),$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_{\frac{n\pi}{\alpha}}(\lambda a)$ and

$$A_{n,m,p} = \frac{4}{a^2 l\alpha\left\{J'_{\frac{n\pi}{\alpha}}(\lambda_p a)\right\}^2}\int_0^{\alpha}\int_{-l}^{l}\int_0^a f(x, \theta, z)x J_{\frac{n\pi}{\alpha}}(\lambda_p x)$$

$$\times \sin\frac{m\pi}{2l}(z+l)\sin\frac{n\pi\theta}{\alpha}\, dx\,dz\,d\theta.$$

7. The temperature of the surface $r = a$ of a solid sphere whose temperature is steady is $v = f(\theta, \phi)$; show that the temperature at any point of the sphere is given by

$$v = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n\int_0^{\pi}\int_0^{2\pi} f(\theta', \phi')P_n(\cos\gamma)\sin\theta'\, d\phi'\,d\theta'.$$

[Use formulae (VII., 41, 42)]

8. Show that, for steady temperature in the cylinder bounded by $r = a$, $z = \pm l$, where $v = f(\theta, z)$ on $r = a$, and $v = 0$ on $z = \pm l$,

$$v = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta)\sin\frac{m\pi}{2l}(z+l)\frac{I_n\left(\frac{m\pi r}{2l}\right)}{I_n\left(\frac{m\pi a}{2l}\right)},$$

where

$$A_{n,m} = \frac{1}{l\pi}\int_0^{2\pi}\int_{-l}^{l} f(\theta, z)\cos n\theta\sin\frac{m\pi}{2l}(z+l)\,dz\,d\theta,$$

π being replaced by 2π when $n = 0$, and $\cos n\theta$ by $\sin n\theta$ in the formula for $B_{n,m}$.

9. Show that the solution for the cylinder of *ex.* 8,

(i) when $v = f(r, \theta)$ on the surface $z = l$, and $v = 0$ on $z = -l$, $r = a$,

(ii) when $v = f(r, \theta)$ on $z = -l$, and $v = 0$ on $z = l$, $r = a$,

is of the form

$$\text{(i)}\quad v = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta)\frac{\sinh\lambda_m(z+l)}{\sinh(2\lambda_m l)}J_n(\lambda_m r),$$

$$\text{(ii)}\quad v = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(C_{n,m}\cos n\theta + D_{n,m}\sin n\theta)\frac{\sinh\lambda_m(l-z)}{\sinh(2\lambda_m l)}J_n(\lambda_m r),$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive zeros of $J_n(\lambda a)$, and find the values of the coefficients.

10. Show that, for the system of spheres

$$x^2 + y^2 + z^2 = \frac{x}{t}$$

with their centres on the x-axis, and touching each other at the origin, a suitable orthogonal system of co-ordinates is given by

$$x = \frac{t}{t^2 + u^2}, \quad y = \frac{u \cos v}{t^2 + u^2}, \quad z = \frac{u \sin v}{t^2 + u^2};$$

and that Laplace's Equation then becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{t^2 + u^2}\frac{\partial V}{\partial t}\right) + \frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u}{t^2 + u^2}\frac{\partial V}{\partial u}\right) + \frac{1}{u^2(t^2 + u^2)}\frac{\partial^2 V}{\partial v^2} = 0,$$

or, if $V = V_1\sqrt{(t^2 + u^2)}$,

$$\frac{\partial^2 V_1}{\partial t^2} + \frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial V_1}{\partial u}\right) + \frac{1}{u^2}\frac{\partial^2 V_1}{\partial v^2} = 0.$$

Deduce that, if $V_1 = e^{pt}(A \cos nv + B \sin nv)\, U$, where U is a function of u only, then

$$u^2\frac{d^2U}{du^2} + u\frac{dU}{du} + (p^2u^2 - n)U = 0.$$

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